Master in Actuarial Science

Models in Finance

07-01-2014

Time allowed: Two and a half hours (150 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider that the share price of a non-dividend paying security is given by a stochastic process S_t which is the solution of the Stochastic Differential Equation (SDE)

$$dS_t = \alpha(t, S_t)\,dt + \sigma(t, S_t)\,dB_t,$$

where:

- B_t is a standard Brownian motion,
- $\alpha(t, x)$ and $\sigma(t, x)$ are differentiable functions with continuous and bounded partial derivatives.
- t is the time from now measured in years.

(a) Consider the process $Y_t = g(t, S_t)$, where $g : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is a function of class $C^{1,2}\left(\mathbb{R}_0^+ \times \mathbb{R}\right)$ such that

$$\frac{\partial g}{\partial t}(t,x) + \frac{\partial g}{\partial x}(t,x)\alpha(t,x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x)\left(\sigma(t,x)\right)^2 = 0,$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Show that

$$dg(t, S_t) = \frac{\partial g}{\partial x}(t, S_t)\sigma(t, S_t)\,dB_t.$$

(10)

(b) Let $\alpha(t, S_t) = 0.05S_t$ and $\sigma(t, S_t) = 0.15S_t$. Calculate the probability that the 5-year return will be at least 20%. (15)

2. Consider the Wilkie model and the equation for the force of inflation

$$I(t) = QMU + QA\left[I(t-1) - QMU\right] + QSD.QZ(t).$$

Assume that the force of inflation over the past year was 0.06 and that the parameter values are:

- $QMU = 0.04$
- $QA = 0.5$
- $QSD = 0.045$

(a) Interpret the meaning of the parameters QMU, QA and QSD? (9)

(b) Calculate the 90% confidence interval for the force of inflation over the next year. (12)

(c) List four financial/economical variables that should be modelled by auto-regressive models, with a mean-reversion effect and explain what are the main differences between these kind of models and random walk processes with positive drift. (14)

3. Consider put and call options written on the same non-dividend paying share with price process S_t and with the same expiry date T and the same exercise price K.

   (a) Show that for the European put option price, we have that (do not use the put-call parity relationship, just use an appropriate portfolio)
   $$p_t \geq Ke^{-r(T-t)} - S_t,$$
   $0 \leq t \leq T$. (12)

   (b) Assume that the current price of the share is 16.5€, the call option price is 1.2€, the put option price is 0.9€, the exercise price is 17€ and the continuously compounded risk-free interest rate is 4% p.a. Calculate the time to expiry of the options. (11)

   (c) What can you say to an investor that wants to exercise an American call option on a non-dividend paying share before the expiry date. Explain your reasons. (12)

4. Consider a 3-period binomial model for the non-dividend paying share with price process S_t such that over each time period the stock price can either move up by 10% or move down by 8%. Assume that the (continuously compounded) risk-free interest rate is 5% per period and that $S_0 = 10$€.

   (a) Construct the binomial tree for the 3-period model and verify if the model is arbitrage free. (14)

   (b) Calculate the price of a derivative with maturity date in 3 periods and with payoff $\max\left\{S_T^2 - K, 0\right\}$, where T is the maturity date, $K = 130$€. Assume that $S_0 = 5$€. (18)

5. Consider a portfolio of 20000 European put options written on a share and N shares. Assume that the delta of an individual option is -0.20.

   (a) Explain what are the steps in the 5-step method which can be used to solve the problems of pricing and hedging of derivatives. (15)

(b) If the portfolio has a delta of zero, calculate the number N of shares in the portfolio. (10)

(c) Consider the Black-Scholes model and a put option written on a dividend paying share with expiry date 9 months from now, strike price 50€ and current price 45€. Assume that the (continuously compounded) free-risk interest rate is 6% p.a., the volatility is 0.15 and the dividends are payable continuously at the constant rate of 2% p.a. Calculate the price of this option. (12)

6. Consider the zero-coupon bond market.

(a) List the desirable characteristics of a term structure model. (16)

(b) Under the real-world probability measure $\mathbb{P}$, the price of a zero-coupon bond with maturity T is

$$B(t,T) = \exp\left\{-(T-t)r\,(t) + \frac{\sigma^2}{6}\,(T-t)^3\right\},$$

where $r\,(t)$ is the short rate of interest at time t.Derive formulas for the instantaneous forward rate $f(t,T)$, the spot rate $R(t,T)$ and the market price of risk $\gamma\,(t,T)$ in terms of $r\,(t)$. In order to derive the formula for $\gamma\,(t,T)$ assume that

$$dr\,(t) = \alpha r\,(t)\,dt + \sigma dZ_t,$$

where $\alpha > 0$ and Z_t is a standard Brownian motion under $\mathbb{P}$. (20)