Master in Actuarial Science

Models in Finance

29-01-2014
Time allowed: Two and a half hours (150 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider that the discounted share price of a non-dividend paying security is given by the stochastic process

$$\widetilde{S}_t = \exp\left\{-\sigma B_t - \left(\alpha^2 + \sigma^2\right) t\right\}$$

where:

- B_t is a standard Brownian motion under the real world measure P,
- α and σ are positive constants.

(a) Deduce the stochastic differential equation (SDE) satisfied by $\widetilde{S}_t$. (10)

(b) The process $\widetilde{S}_t$ is a martingale under real world measure P? And under the equivalent martingale measure or risk neutral measure Q, what would be the SDE satisfied by $\widetilde{S}_t$? (10)

2. Consider the continuous time lognormal model for the market price of an investment S_t.

(a) What is the distribution of the log returns $\log(S_u) - \log(S_t)$, for $u > t$ and what is the expected value $\mathbb{E}[S_u]$ and the variance $\mathrm{Var}[S_u]$ if S_t is known and the lognormal model drift parameter is μ and its volatility parameter is σ. (10)

(b) What are the advantages and disadvantages of the normal distribution assumption of the lognormal model and what kind of models can be used in order to obtain non-normal distributions? (12)

(c) Considering time series models of financial markets, discuss the difference between a cross sectional property and a longitudinal property. Discuss also the difference between these properties in a random walk environment. (15)

3. Consider European put and call options written on a dividend paying share.

(a) Explain how and why the time to expiry, the interest rates and the dividend income received on the underlying security affect the price of the European call and put options. (15)

(b) By constructing two portfolios with identical payoffs at the exercise date of the options, derive an expression for the put-call parity of European options on a dividend paying share, where the dividend H is known to be payable at some date t^* with $t < t^* < T$. More precisely, prove that

$$c_t + He^{-r(t^*-t)} + Ke^{-r(T-t)} = p_t + S_t.$$

(16)

(c) Consider a call option with price (at time t) given by $c_t = 0.8$, a put option with price $p_t = 0.6$, written on the same underlying share, with time to maturity 15 months and the same strike price 25€. Assume that the current share price is 20€, the continuously compounded risk-free interest rate is 7% p.a. and the share pays a dividend of 1€ at a date 3 months before maturity. Is the put-call relationship satisfied? What can you say about the model used to calculate c_t and p_t? (10)

4. Consider a binomial model for the non-dividend paying share with price process S_t such that the price at time $t+1$ is either $1.15S_t$ or $0.9S_t$ (assume that the time t is measured in years). Assume that the continuously compounded risk-free interest rate is 10% p.a. and that $S_0 = 10$. Consider a derivative D with payoff at time $t = 2$ given by

$$\begin{aligned} c_2(1) &= S_2 - 3.225 \quad \text{if } S_2 = S_0u^2, \\ c_2(2) &= S_2 - 5.35 \quad \text{if } S_2 = S_0ud, \\ c_2(3) &= 0 \quad \text{if } S_2 = S_0d^2, \end{aligned}$$

where u and d are the sizes of the up-step and down-step in each period.

(a) Calculate the risk-neutral probability measure and construct the binomial tree. (14)

(b) Calculate the price of the derivative D at time $t = 0$ and describe how you could derive the hedging strategy (i.e., state a general formula for the portfolio composed of the underlying security and the risk free asset required to hedge the derivative security). (18)

5. Consider the Black-Scholes model and a European call option written on a non-dividend paying share with expiry date 15 months from now,

strike price 30€ and current price 25€. Assume that the (continuously compounded) free-risk interest rate is 8% p.a. and that the volatility is $\sigma = 0.2$.

(a) Define the greeks Delta (Δ) and vega (ν) for a general derivative and calculate the delta for the call option (considering the Black-Scholes model). (12)

(b) Consider that an investor has 10000 call options as defined above. Calculate the corresponding hedging portfolio in shares and cash. (14)

(c) Consider a derivative Φ that has the following payoff at expiry date T depending on the price of the underlying non-dividend paying share at maturity T and at a previous time $T_0 < T$:

$$Payoff = \frac{S(T)}{S(T_0)}.$$

Show that the price of the derivative at time t is given by (for $t < T_0 < T$)

$$e^{-r(T-t)}e^{\left(r-\frac{\sigma^2}{2}\right)(T-T_0)}E_{t,s}^{Q}\left[\exp\left(\sigma\left(Z_T - Z_{T_0}\right)\right)\right] = e^{-r(T_0-t)},$$

where Z is a standard Brownian motion with respect to the measure Q. (18)

6. Consider the zero-coupon bond market.

(a) Present the stochastic differential equations (SDE), under the risk neutral measure Q, for the short rate in the Hull-White model and in the 2-factor Vacisek model, defining all the notation used. (12)

(b) Discuss the main differences and advantages/disadvantages between the Hull-White model and the one-factor Vacisek model (14)