Master in Actuarial Science

Models in Finance
29-01-2014

Solutions

1. (a) By Itô's lemma (or Itô's formula) applied to $f(t,x) = \exp(-\sigma x - (\alpha^2+\sigma^2)t)$ (it is a $C^{1,2}$ function):

$$\begin{aligned} d\widetilde{S}_t &= \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)\left(dB_t\right)^2 \\ &= -(\alpha^2+\sigma^2)\widetilde{S}_t dt - \sigma \widetilde{S}_t dB_t + \frac{1}{2}\sigma^2 \widetilde{S}_t dt \\ &= -\left(\alpha^2 + \frac{\sigma^2}{2}\right)\widetilde{S}_t dt - \sigma \widetilde{S}_t dB_t. \end{aligned}$$

where we have used $\left(dB_t\right)^2 = dt$. Therefore

$$d\widetilde{S}_t = -\left(\alpha^2 + \frac{\sigma^2}{2}\right)\widetilde{S}_t dt - \sigma \widetilde{S}_t dB_t.$$

(b) In general, the discounted price process $\widetilde{S}_t$ is not a martingale under the real world probability $\mathbb{P}$. Indeed, since in the SDE above, the drift coefficient $-\left(\alpha^2 + \frac{\sigma^2}{2}\right)\widetilde{S}_t$ is not zero, the process $\widetilde{S}_t$ is not a martingale.

Under the equivalent martingale measure $Q$, the discounted price process $\widetilde{S}_t$ is a martingale, the drift coefficient is zero and the diffusion coefficient of the SDE remains the same, i.e.

$$d\widetilde{S}_t = -\sigma \widetilde{S}_t d\overline{B}_t,$$

where $\overline{B}_t$ is a standard Brownian motion under $Q$.

2. (a) The log returns have a normal distribution:

$$\log\left(S_u\right) - \log\left(S_t\right) \sim N\left[\mu\left(u-t\right), \sigma^2\left(u-t\right)\right],$$

and

$$E[S_u] = S_t \exp\left(\mu(u-t) + \frac{1}{2}\sigma^2(u-t)\right),$$

$$Var[S_u] = S_t^2 \exp\left(2\mu(u-t) + \sigma^2(u-t)\right)\left[\exp\left(\sigma^2(u-t)\right) - 1\right].$$

(b) Normality assumption: market crashes appear more often than one would expect from a normal distribution of the log-returns (the empirical distribution has "fat tails" when compared to the Normal). Moreover, days with very small changes also happen more often than the normal distribution suggests (more peaked distribution). The main advantage of considering the normal distribution is its mathematical tractability.

The fat tails and jumps justify the consideration of Lévy processes (associated with "fat tails") for modelling security prices.

(c) A cross-sectional property fixes a time horizon and looks at the distribution over all the simulations. For example, we might consider the distribution of inflation next year. Implicitly, this is a distribution conditional on the past information which is built into the initial conditions and is common to all simulations. If those initial conditions change, then the implied cross-sectional distribution will also change.

A longitudinal property picks one simulation and looks at a statistic sampled repeatedly from that simulation over a long period of time. Unlike cross-sectional properties, longitudinal properties do not reflect market conditions at a particular date but, rather, an average over all likely future economic conditions.

In a pure random walk environment, asset returns are independent across years and also (as for any model) across simulations. As a result, cross-sectional and longitudinal quantities coincide. To equate the two is valid in a random walk setting, but not for more general models.

3. (a) The longer the time to expiry, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with term to maturity.

Interest rates:

Call option: an increase in the risk-free rate of interest will result in a higher value for the option because the money saved by purchasing the option rather than the underlying share can be invested at this higher rate of interest, thus increasing the value of the option.

Put option: higher interest means a lower value (put options can be purchased as a way of deferring the sale of a share: the money is tied up for longer)

Dividend income:

Call option: the higher the level of dividend income received, the lower is the value of a call option, because by buying the option instead of the underlying share the investor loses this income.

Put option: the higher the level of dividend income received, the higher is the value of a put option, because buying the option is a way of deferring the sale of a share and the dividend income is received.

(b) Let us consider two portfolios. Portfolio $A$: one European call option + cash $He^{-r(t^*-t)} + Ke^{-r(T-t)}$

Portfolio $B$: one European put option + one dividend paying share.

At time $T$, the value of portfolio $A$ is $S_T - K + He^{r(T-t^*)} + K = S_T + He^{r(T-t^*)}$ if $S_T > K$ and $He^{r(T-t^*)} + K$ if $S_T \leq K$.

At time $T$, the value of portfolio $B$ is $0+S_T+He^{r(T-t^*)}$ if $S_T > K$ and $K - S_T + S_T + He^{r(T-t^*)} = He^{r(T-t^*)} + K$ if $S_T \leq K$.

Therefore, the portfolios have the same value at maturity. Then, by the no-arbitrage principle, the porfolios have the same value for any time $t < T$, i.e.,

$$c_t + He^{-r(t^*-t)} + Ke^{-r(T-t)} = p_t + S_t.$$

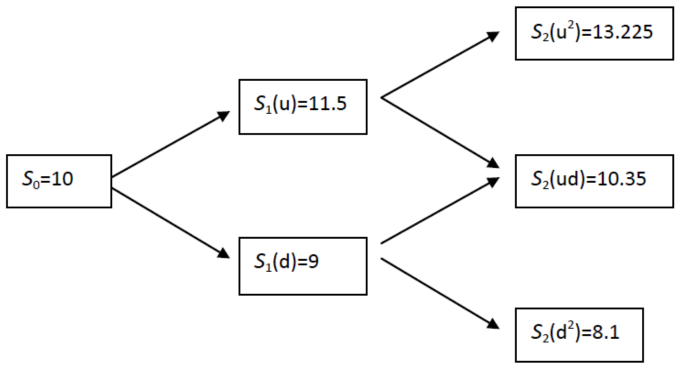
(c) We have that

$$\begin{aligned} c_t + He^{-r(t^*-t)} + Ke^{-r(T-t)} &= 0.8 + e^{-0.07\times 1} + 25e^{-0.07\times\left(\frac{15}{12}\right)} \\ &= 24.638 \end{aligned}$$

and

$$p_t + S_t = 0.6 + 20 = 20.6$$

Therefore, the put-call relationship is not satisfied. This means that the model used to calculate the prices of the options is not arbitrage free.




4. (a) $\frac{S_{t+1}}{S_t} = 1.15$ or $\frac{S_{t+1}}{S_t} = 0.9$. Therefore $u = 1.15$ and $d = 0.9$.
$e^r = e^{0.10} = 1.1052$ and we have $d < e^r < u$ and therefore the model is arbitrage free.
The risk neutral probability of an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.10} - 0.9}{1.15 - 0.9} = 0.8207.$$

Binomial tree:

(b) Payoff of the derivative: $C_2\left(u^2\right) = S_2 - 3.225 = 10$, $C_2\left(ud\right) = S_2 - 5.35 = 5$, $C_2\left(d^2\right) = 0$.
Using the usual backward procedure:
At time 1: $C_1\left(u\right) = \exp(-r)\left[qC_2\left(u^2\right) + (1-q)C_2\left(ud\right)\right] = 8.2372$,
$C_1\left(d\right) = \exp(-r)\left[qC_2\left(ud^2\right) + (1-q)C_2\left(d^2\right)\right] = 3.713$
At time 0, the price is $C_0 = \exp(-r)\left[qC_1\left(u\right) + (1-q)C_1\left(d\right)\right] = 6.7193$.
In order to calculate the hedging strategy, we could use the formulas (generalization of the formulas for the one-period model): for time $t$ and state $j$, we should apply the formulas

$$\phi_{t+1}\left(j\right) = \frac{C_{t+1}\left(ju\right) - C_{t+1}\left(jd\right)}{S_t\left(j\right)\left(u-d\right)},$$
$$\psi_{t+1}\left(j\right) = e^{-r}\left[\frac{C_{t+1}\left(jd\right)u - C_2\left(ju\right)d}{u-d}\right].$$

where $\phi$ represents the units of stock in the portfolio and $\psi$ represents the units of cash.

5. (a) Let $f(t,s)$ be the value at time $t$ of a derivative when the price of the the underlying asset at $t$ is $S_t = s$.

Delta of the derivative and vega:

$$\Delta = \frac{\partial f}{\partial s},$$
$$\nu = \frac{\partial f}{\partial \sigma}.$$

Vega is the rate of change of the price of the derivative with respect to a change in the volatility of $S_t$.

The delta of a call option can be derived from the Black-Scholes formula and is given by $\Delta = \frac{\partial c_t}{\partial S_t} = \Phi(d_1)$, where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = -0.2564$$

and $\Delta = \Phi(-0.2564) = 0.3988$.

(b) The option price is given by:

$$c_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2) = 1.4032$$

where $d_1 = -0.2564$ and $d_2 = d_1 - \sigma\sqrt{T-t} = -0.48$. Hence,

The hedging portfolio is: $\Delta \times$number of options $= 0.3988 \times 10000 = 3988$ units of stock and $10000 \times 1.4032 - 3988 \times 25 = -85668$€ in cash.

(c) The dynamics of the stock prices $S_t$ under $Q$ is given by the SDE

$$dS_u = r\, S_u\, du + \sigma\, S_u\, dZ_u,$$
$$S_t = s$$

This is a geometric Brownian motion and the solution is such that:

$$S_T = s \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(Z_T - Z_t\right)\right],$$
$$S_{T_0} = s \exp\left[\left(r - \frac{\sigma^2}{2}\right)(T_0-t) + \sigma\left(Z_{T_0} - Z_t\right)\right]$$

The price of the derivative is given by

$$\begin{aligned} F(t, S_t) &= e^{-r(T-t)} E_{t,s}^{Q}\left[\frac{S_T}{S_{T_0}}\right] \\ &= e^{-r(T-t)} e^{\left(r-\frac{\sigma^2}{2}\right)(T-T_0)} E_{t,s}^{Q}\left[\exp\left(\sigma\left(Z_T - Z_{T_0}\right)\right)\right] \\ &= e^{-r(T-t)} e^{\left(r-\frac{\sigma^2}{2}\right)(T-T_0)} e^{\frac{1}{2}\sigma^2(T-T_0)} = e^{-r(T_0-t)}. \end{aligned}$$

6. (a) The Hull-White model SDE for the short rate $r(t)$ under $Q$:

$$dr(t) = \alpha\left(\mu(t) - r(t)\right) dt + \sigma d\widetilde{W}_t,$$

where $\widetilde{W}_t$ is a standard Bm under $Q$, the parameter $\alpha$ is positive and $\mu(t)$ is a deterministic function.

In the 2-factor Vasicek model there are two processes: $r(t)$ and $m(t)$, the local mean reversion level:

$$\begin{aligned} dr(t) &= \alpha_r\left(m(t) - r(t)\right) dt + \sigma_{r1} d\widetilde{W}_1(t) + \sigma_{r2} d\widetilde{W}_2(t), \\ dm(t) &= \alpha_m\left(\mu - m(t)\right) dt + \sigma_{m1} d\widetilde{W}_1(t), \end{aligned}$$

where $\widetilde{W}_1(t)$ and $\widetilde{W}_2(t)$ are independent, standard Brownian motions under the risk neutral measure $Q$.

(b) The SDEs for the Vasicek model gives us a time-homogeneous model. This implies lack of flexibility for pricing related contracts. A simple way to get theoretical prices to match observed market prices is to introduce some elements of time-inhomogeneity into the model. The Hull & White (HW) model does this. This model is similar to Vasicek model but now $\mu(t)$ is no longer a constant. The HW model can even be extended to include a time-varying deterministic $\sigma(t)$. This allows us to calibrate the model to traded option prices as well as zero-coupon bond prices. Moreover, since $\mu(t)$ is deterministic, the HW model is as tractable as the Vasicek model.

The HW model suffers from the same drawback as the Vasicek model: interest rates might become negative.