Lévy models with stochastic volatility

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Introduction

- Empirical data shows that volatility changes stochastically over time: it has been observed that the estimated volatilities change stochastically over time and are clustered.
- Two main methods to incorporate the stochastic volatility effect:
- First method: the volatility parameter of the Black–Scholes model can be made stochastic by following an appropriate SDE. This technique was implemented by Hull and White (1988) and Heston(1993). In these models, the volatility process is driven by a Brownian motion.However, one can consider that the volatility is driven by OU (Ornstein-Uhlenbeck) process driven by a suitable subordinator (non decreasing Lévy process). These models were considered by Barndorff-Nielsen and Shephard (Barndorff-Nielsen and Shephard (2001, 2003)).

- Second method: We can increase or decrease the volatility by speeding up or slowing down the rate at which time passes. In order to build clustering and to keep time going forward, employ a mean-reverting positive process as a measure of the local rate of time change.
- Basic intuition underlying the Second method: the Brownian motion self-similarity property relates changes in scale to changes in time $\Longrightarrow$ random changes in volatility can be captured by random changes in time.
- Some candidates for the rate of time change: OU process or the classical CIR process.
- The Second method was proposed by Carr, Madan, Geman and Yor (2003).

Lévy Models with stochastic volatility

The BNS model

- Consider that the volatility follows an OU process driven by a subordinator. These models are called BNS models after Barndorff-Nielsen and Shephard.An alternative name is Black–Scholes-SV models.
- Consider the SDE for the stock price in the B-S model:

$$dS_t = S_t \left(\mu dt + \sigma d\overline{W}_t \right). \tag{1}$$

- Under the transformation $L_t = \log(S_t)$, applying the Itô formula, we obtain the SDE for the log-price L:

$$dL_t = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma d\overline{W}_t. \tag{2}$$

- Assume that σ_t^2 follows a OU process:

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + d\overline{z}_{\lambda t}, \tag{3}$$

where $\overline{z}_t$ is a subordinator (non decreasing Lévy process).We assume that $\overline{z}_t$ has no drift.

- Under a technical assumption on the existence of exponential moments for the process $\overline{z}_t$, it is possible to show that:

$$dL_t = \left(\mu - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t d\overline{W}_t + \rho d\overline{z}_{\lambda t}, \tag{4}$$

where $\rho \leq 0$ (represents the correlation between S_t and σ_t^2).

- It is often observed that a down-jump in the stock-price corresponds to an up-jump in volatility $\Longrightarrow \rho \leq 0$.

- The BM $\overline{W}_t$ and $\overline{z}_t$ are independent.
- The model is arbitrage free but incomplete (exists more than one equivalent martingale measure).
- The SDE's of L_t and σ_t^2 under the equivalent martingale measure Q is:

$$dL_t = \left(r - q - \lambda k(-\rho) - \frac{1}{2}\sigma_t^2\right) dt + \sigma_t dW_t + \rho dz_{\lambda t},$$
$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dz_{\lambda t},$$

where W_t is a Brownian motion under Q and independent of the subordinator z_t, and $k(u) = k^Q(u) = \log\left[E_Q\left[\exp(-uz_1)\right]\right]$ is the cumulant function of z_1 under Q.

BNS model with Gamma SV

- The BNS model with Gamma Stochastic volatility (BNS OU-Gamma SV):
- In this model, the volatility σ_t^2 follows the OU process (3) with a marginal (stationary) Gamma(a, b) distribution.
- The process z_t has a Lévy density $w(x) = ab\exp(-bx)$and the associated cumulant function is

$$k(u) = -au(b+u)^{-1}.$$

BNS model with IG SV

- The BNS model with Inverse Gaussian (IG) stochastic volatility (BNS OU-IG): in this model, the volatility σ_t^2 follows the OU process (3) with a marginal (stationary) IG(a, b) distribution.
- The associated cumulant function is

$$k(u) = -uab^{-1}(1+2ub^{-2})^{-1/2}.$$

The Stochastic time change

- We now consider the stochastic time approach.
- In periods of high volatility, time will run faster than in periods of low volatility.
- The first application of stochastic time change to asset pricing was by Clark (1973), who modelled the asset price as a geometric Brownian motion subordinated by an independent Lévy subordinator.

- Since time needs to increase, all processes modelling the rate of time change need to be positive.
- The first candidate is the classical mean-reverting CIR process, which is based on Brownian motion.
- A second group of candidates is the OU processes driven by a subordinator, like the Gamma–OU process or the IG–OU process.

The Integrated CIR time change

- Mean-reverting positive stochastic process: the Cox–Ingersoll–Ross (CIR) process y_t, solution of the SDE

$$dy_t = k\,(\eta - y_t)\,dt + \lambda y_t^{1/2} dW_t,$$

  where W_t is a standard Brownian motion.
- η represents the long run rate of time change, k is the rate of mean reversion and λ is related to the volatility of time change.
- The mean and variance of y_t, given y_0, are

$$\mathbb{E}\,[y_t|y_0] = y_0 \exp(-kt) + \eta\,(1 - \exp(-kt))\,,$$
$$\mathrm{var}[y_t|y_0] = y_0 \frac{\lambda^2}{k}\,(\exp(-kt) - \exp(-2kt)) + \frac{\eta\lambda^2}{2k}\,(1 - \exp(-kt))^2$$

The integrated CIR time change

- The economic time elapsed in t units of calendar time is given by the integrated CIR process Y_t, defined by

$$Y_t = \int_0^t y_s ds,$$

  where y_t is the CIR process.
- Since y_t is a positive process $\Longrightarrow$ Y_t is an increasing process.
- The characteristic function of Y_t is explicitly known as a function of k, η, λ, y_0 and t and from the characteristic function it is possible to deduce that

$$\mathbb{E}\,[Y_t|y_0] = \eta t + k^{-1}\,(y_0 - \eta)\,(1 - \exp(-kt))\,.$$

The IntOU time change

- For the rate of time change we can also use an OU process driven by a subordinator.
- The rate of time change y_t, is the solution of the SDE

$$dy_t = -\lambda y_t dt + dz_{\lambda t},$$

where z_t is a subordinator (a non decreasing Lévy process).
- The intOU process is $Y_t = \int_0^t y_s ds$.
- For the subordinator, one can use a Gamma distribution such that y_t has a Gamma(a, b) stationary distribution and we have a Gamma-OU model.
- We can also use a IG (Inverse Gaussian) distribution such that y_t has a IG(a, b) stationary distribution and we have a IG-OU model.

The Lévy SV market model

- Let Y_t be the process we choose to model our business time (time change).
- The risk neutral price process (under the measure Q) can be modelled by

$$S_t = S_0 \frac{\exp((r-q)t)}{\mathbb{E}\left[\exp\left(X_{Y_t}\right)|y_0\right]} \exp\left(X_{Y_t}\right),$$

where X is a Lévy process.
- The factor $\frac{\exp((r-q)t)}{\mathbb{E}\left[\exp\left(X_{Y_t}\right)|y_0\right]}$ is such that the discounted price process is a Q-martingale.
- The process incorporates jumps (through the Lévy process X) and stochastic volatility (through the time change Y_t).
- In general terms, stochastic volatility is needed to explain the variation in strike of option prices at longer terms, while jumps are needed to explain the variation in strike at shorter terms.

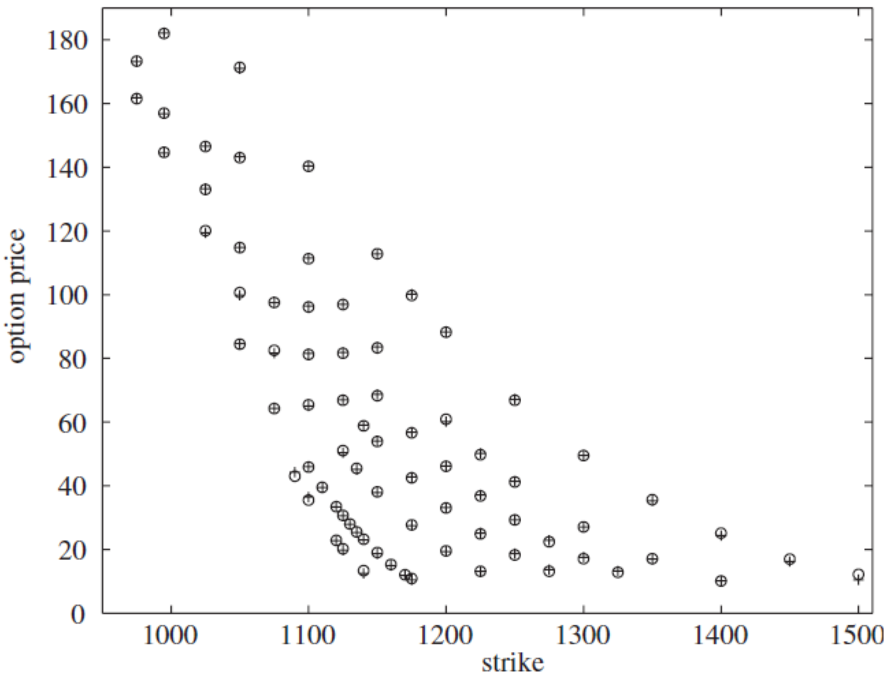

Figure 7.4 Meixner–OU–Gamma calibration of S&P 500 options (circles are market prices, pluses are model prices).

The Lévy SV market model

Table 7.4 APE, AAE, RMSE and ARPE for Lévy SV models.

Model	APE (%)	AAE	RMSE	ARPE (%)
CGMY–CIR	0.56	0.3483	0.4367	1.15
CGMY–Gamma–OU	0.42	0.2576	0.3646	0.90
CGMY–IG–OU	0.44	0.2728	0.3736	0.90
VG–CIR	0.69	0.4269	0.5003	1.33
VG–Gamma–OU	0.51	0.3171	0.4393	1.10
VG–IG–OU	0.52	0.3188	0.4306	1.05
NIG–CIR	0.67	0.4123	0.4814	1.32
NIG–Gamma–OU	0.58	0.3559	0.4510	1.27
NIG–IG–OU	0.53	0.3277	0.4156	1.05
Meixner–CIR	0.68	0.4204	0.4896	1.34
Meixner–Gamma–OU	0.49	0.3033	0.4180	1.06
Meixner–IG–OU	0.50	0.3090	0.4140	1.03
GH–CIR	0.65	0.4032	0.4724	1.30
GH–Gamma–OU	0.45	0.2782	0.3837	0.95
GH–IG–OU	0.49	0.3041	0.3881	1.01

- Cont, R. and Tankov, P. (2003). Financial modelling with jump processes. Chapman and Hall/CRC Press.
- Schoutens, W. (2002). Lévy Processes in Finance: Pricing Financial Derivatives. Wiley.