Master in Actuarial Science

Models in Finance

06-01-2015

Time allowed: Two and a half hours (150 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Ley Y_t be a stochastic process which is the solution of the Stochastic Differential Equation (SDE)

$$dY_t = e^{-t} Y_t dt + \sigma Y_t dB_t,$$

where B_t is a standard Brownian motion and σ is a constant.

   (a) Solve the stochastic differential equation and calculate $\mathbb{E}[Y_t]$. (15)

   (b) Consider that Y_t is the share price of a non-dividend paying security. Let $\sigma = 0.2$ and consider the time variable is measured in years. Calculate the probability that, over a 3 year period, the share price will fall at least 5%. (12)

2. Consider the continuous time lognormal model of security prices.

   (a) Discuss the empirical evidence and the theoretical arguments about the behaviour of stock prices with respect to: (i) the time evolution of the expected value of the returns; (ii) the time evolution of the volatilty of returns; (iii) the mean reverting effect of the stock prices; (iv) the distribution of returns. (16)

   (b) Discuss how well the continuous time lognormal model describes the empirical features of part (a). (12)

   (c) What type of alternative models (alternatives to the lognormal model) can be considered in order to satisfy the empirical features of part (a)? (8)

3. Consider a share with price process S_t and a forward contract written on S with maturity date T. Assume that the share pays dividends at a constant rate (continuously compounded) q. The risk-free interest rate is assumed to be r (constant).

   (a) Deduce a formula for the (fair) forward price K of the forward contract, at time $t = 0$. (16)

   (b) If the forward price is 30€, the current price is 25€, the risk-free interest rate is 10% p.a. (continuously compounded) and the time to expiry is 30 months, calculate the dividend rate of the share. (10)

4. Consider a 3-period recombining binomial model for the non-dividend paying share with price process S_t such that over each time period the stock price can either move up by 10% or move down by 5%. Assume that the current price of the share is 4€.

(a) Discuss if the model is free of arbitrage (if it is not, prove that there is an arbitrage strategy) in the following cases:

   i. if the risk-free interest rate is 12% per period (continuously compounded).

   ii. if the risk-free interest rate is 1% per period (continuously compounded). (12)

(b) If the continuously compounded risk-free interest rate is 4% per period, calculate the risk neutral probability measure and construct the binomial tree. (12)

(c) Calculate the price of a derivative with maturity T =3 periods, interest rate as in (b) and with payoff

$$\max\left\{90 - \exp\left(S_T\right), 0\right\}.$$

(16)

5. Consider the Black-Scholes model.

   (a) List the assumptions underlying the Black-Scholes model and state the general risk-neutral valuation formula for the price, at time $t < T$, of a derivative security with payoff X at the expiry date T. (15)

   (b) Consider a financial derivative with the payoff

$$X = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S_u du,$$

   where T_2 is the maturity date, $T_1 < T_2$, and S_u is the price of the underlying non-dividend paying share at time u. Show that the price of this financial derivative at time $t < T_1$ is given by

$$V_t = \frac{S_t}{r\left(T_2 - T_1\right)}\left[1 - \exp\left(-r\left(T_2 - T_1\right)\right)\right].$$

(18)

6. Consider the zero-coupon bond market for zero-coupon bonds paying 1€ at time T.

   (a) Discuss the limitations of one-factor interest rate models. (15)

(b) Consider that the instantaneous forward rate is given by

$$f(t,T) = r(t) - \alpha\,(T-t)^4\,,$$

where $\alpha > 0$ is a constant and $r(t)$ is the short rate. Calculate:

i. the price of the zero-coupon bond at time t.

ii. the spot rate curve at time t. (15)

(c) Calculate the price of a 2-year zero coupon bond, assuming that $r(t) = 0.2$ (constant) and $\alpha = 0.01$. (8)