Master in Actuarial Science

Models in Finance
06-01-2015

Solutions

1. (a) Define $Z_t = \ln(Y_t)$. By Itô's lemma (or Itô's formula) applied to $f(x) = \ln(x)$:

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial x}(Y_t) dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, Y_t)(dY_t)^2 \\ &= e^{-t}dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \\ &= \left(e^{-t} - \frac{1}{2}\sigma^2\right) dt + \sigma dB_t. \end{aligned}$$

where we have used $(dB_t)^2 = dt$. Therefore

$$Z_t = \ln(Y_0) + \int_0^t \left(e^{-s} - \frac{1}{2}\sigma^2\right) ds + \sigma B_t,$$

and

$$Y_t = Y_0 \exp\left(1 - e^{-t} - \frac{1}{2}\sigma^2 t + \sigma B_t\right).$$

Moreover,

$$\mathbb{E}[Y_t] = Y_0 \exp\left(1 - e^{-t}\right).$$

(b) We have

$$Y_3 = Y_0 \exp(0.8902 + 0.2B_3)$$

Since $B_t \sim N(0; t)$, then $V := \log\left(\frac{Y_3}{Y_0}\right) \sim N(0.8902; 0.12)$, we have

$$\begin{aligned} P\left(\frac{Y_3}{Y_0} \leq 0.95\right) &= P(\exp(0.8902 + 0.2B_3) \leq 0.95) \\ &= P(V \leq \ln(0.95)) = 0.0033. \end{aligned}$$

2. (a) (i) There are good theoretical reasons to suppose that the expected returns per time unit should vary over time. It is reasonable to suppose that investors will require a risk premium on equities relative to bonds. As a result, if interest rates are high, we might expect the expected value of returns to be high as well. However, it is not easy to test this argument empirically.

(ii) Empirical data shows that volatility parameter is not constant in time. The implied volatility obtained from option prices and the examination of historic option prices suggests that volatility expectations fluctuate markedly over time.

(iii) One unsettled empirical question is whether markets are mean reverting, or not. A mean reverting market is one where rises are more likely following a market fall, and falls are more likely following a rise. There appears to be some evidence for this, but the evidence rests heavily on the aftermath of a small number of dramatic crashes. Furthermore, there also appears to be some evidence of momentum effects, which imply that a rise one day is more likely to be followed by another rise the next day.

(iv) About the use of the normality assumptions in market returns: market crashes appear more often than one would expect from a normal distribution. Empirical data shows that extreme events occur more often (distribution with "heavy tails") than in a normal distribution. While the random walk produces continuous price paths, jumps or discontinuities seem to be an important feature of real markets. Furthermore, days with no change, or very small change, also happen more often than the normal distribution suggests.

(b) (i) In the lognormal model, the expected value of returns per time unit, or drift, is constant, which does not agree with the theoretical argument given in (a). However, in this case it is difficult to test empirically if it is really necessary to assume a non-constant drift.

(ii) In the lognormal model, the volatility is assumed to be constant, in contradiction with empirical evidence.

(iii) The lognormal model is not mean reverting. However, there is no strong empirical evidence of mean-reversion effects in stock prices.

(iv) The lognormal model implies normal log-returns. This contradicts the empirical evidence described in (a).

(c) One class of models with the feature of non-constant volatility are the ARCH models. Models with non-normal returns or stochastic volatility models also satisfy this property.

Models based on Lévy processes can also be used in order to obtain non-normal returns.

3. (a) Consider the 2 portfolios:

A: one long position in the forward contract (that gives you a share at time T by the price K). This portfolio has value K at time 0 and value $S_T - K$ at time T.

B: borrow Ke^{-rT} in cash and buy one fraction of the share given by $e^{-qT}S_0$. This portfolio has value $e^{-qT}S_0 - Ke^{-rT}$ at time 0 and value $S_T - K$ at time T (assuming that the dividends are reinvested in the share).

At time T both portfolios have a value of $S_T - K$. By the principle of no arbitrage, these portfolios must have the same value at time 0. Since at time 0, the value of portfolio B is $e^{-qT}S_0 - Ke^{-rT}$ and the value of portfolio A at time 0 is 0 (the value of the forward contract at time 0 must be zero), we have $e^{-qT}S_0 - Ke^{-rT} = 0$ and

$$K = S_0 e^{(r-q)T}.$$

(b) We have that $K = S_0 e^{(r-q)T}$. Therefore $(r-q)\,T = \ln\left(\frac{K}{S_0}\right)$ and

$$q = r - \frac{1}{T}\ln\left(\frac{K}{S_0}\right).$$

Now, with $T = 2.5$ years, $r = 0.1$, $K = 30$ and $S_0 = 25$, we obtain

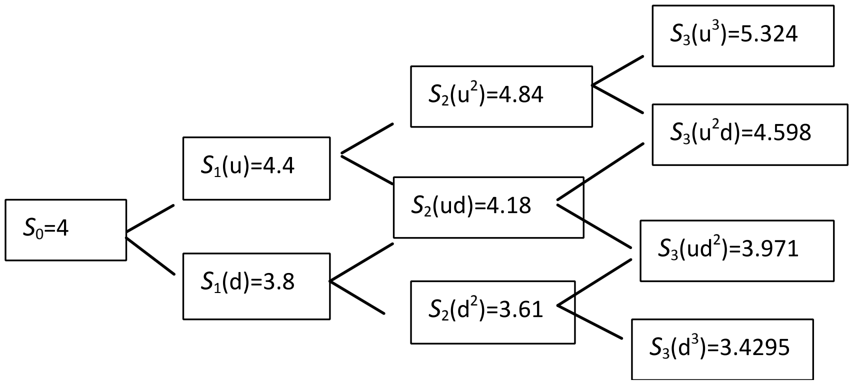
$$q = 0.1 - \frac{1}{2.5}\ln\left(\frac{30}{25}\right) = 0.0271$$

and the annual dividend rate is 2.71%.

4. (a) $u = 1.1$ and $d = 0.95$

(i) The model is arbitrage free if and only if $d < e^r < u$. If $r = 0.12$ then $e^r = 1.1275 > u$.

In this case $u < e^r$ and the cash investment would outperform the share investment in all circumstances. An investor could (at time 0) sell the share and invest $S_0 = 4$ Euros in a cash account. At time 1 he could buy again the share and have a certain positive



profit of $S_0 e^r - S_0 u = 4\exp(0.12) - 4 \times 1.10 = 0.11 > 0$ or $S_0 e^r - S_0 d = 4\exp(0.12) - 4 \times 0.95 = 0.71 > 0$ (arbitrage opportunity).
(ii) If $r = 0.01$ then $e^r = 1.0101$. In this case, $d < e^r < u$ and the model is arbitrage free.

(b) We have $u = 1.10$, $d = 0.95$, $S_0 = 4$ and $r = 0.04$. Therefore $e^r = 1.0408$ and $d < e^r < u$. The model is arbitrage free.
The risk neutral probability of an up-movement is

$$q = \frac{e^r - d}{u - d} = \frac{e^{0.04} - 0.95}{1.1 - 0.95} = 0.6054.$$

Binomial tree:

(c) Payoff of the derivative: $C_3 = \max\{90 - \exp(S_3), 0\}$.
Using the usual backward procedure:
$C_3(u^3) = \max\{90 - \exp(S_0 u^3), 0\} = 0$, $C_3(u^2 d) = \max\{90 - \exp(S_0 u^2 d), 0\} = 0$, $C_3(d^2 u) = \max\{90 - \exp(S_0 d^2 u), 0\} = 36.962$ and $C_3(d^3) = \max\{90 - \exp(S_0 d^3), 0\} = 59.139$.
At time 2: $C_2(u^2) = 0$, $C_2(ud) = \exp(-r)\left[q \times 0 + (1-q) C_3(d^2 u)\right] = 14.013$, $C_2(d^2) = \exp(-r)\left[q C_3(d^2 u) + (1-q) C_3(d^3)\right] = 43.921$.
At time 1: $C_1(u) = \exp(-r)\left[q \times 0 + (1-q) C_2(ud)\right] = 5.3127$, $C_1(d) = \exp(-r)\left[q C_2(ud) + (1-q) C_2(d^2)\right] = 24.802$.
At time 0, the price is $C_0 = \exp(-r)\left[q C_1(u) + (1-q) C_1(d)\right] = 12.493$.

5. (a) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion.

2. There are no risk-free arbitrage opportunities.

3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending.

4. Unlimited short selling (that is, negative holdings) is allowed.

5. There are no taxes or transaction costs.

6. The underlying asset can be traded continuously and in infinitesimally small numbers of units.

The general risk-neutral valuation formula for a derivative with payoff X is

$$V_t = e^{-r(T-t)}\mathbb{E}_Q\left[X|\mathcal{F}_t\right]$$

(b)

$$\begin{aligned} V_t &= e^{-r(T_2-t)}\mathbb{E}_Q\left[\frac{1}{T_2-T_1}\int_{T_1}^{T_2} S_u du|\mathcal{F}_t\right] \\ &= e^{-r(T_2-t)}\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\mathbb{E}_Q\left[S_u|\mathcal{F}_t\right]du \end{aligned}$$

The dynamics of the stock prices S_t under Q is given by the SDE

$$\begin{aligned} dS_u &= r\,S_u\,du + \sigma\,S_u\,dZ_u, \quad u > t \\ S_t &= s \end{aligned}$$

This is a geometric Brownian motion and the solution is such that:

$$S_u = s\exp\left[\left(r - \frac{\sigma^2}{2}\right)(u-t) + \sigma\left(Z_u - Z_t\right)\right],$$

and

$$\mathbb{E}_Q\left[S_u|\mathcal{F}_t\right] = \mathbb{E}_Q\left[S_u|S_t\right] = S_t e^{r(u-t)}.$$

Therefore

$$\begin{aligned} V_t &= \frac{e^{-r(T_2-t)}}{T_2-T_1}\int_{T_1}^{T_2} S_t e^{r(u-t)}du \\ &= \frac{e^{-r(T_2-t)}S_t}{(T_2-T_1)\,r}\left[e^{r(T_2-t)} - e^{r(T_1-t)}\right] \\ &= \frac{S_t}{r\left(T_2-T_1\right)}\left[1-\exp\left(-r\left(T_2-T_1\right)\right)\right]. \end{aligned}$$

6. (a) (1) if we look at historical interest rate data we can see that changes in the prices of bonds with different terms to maturity are not perfectly correlated as one would expect to see if a one-factor model was correct. Sometimes we even see, for example, that short-dated bonds fall in price while long-dated bonds go up.

(2) If we look at the long run of historical data we find that there have been sustained periods of both high and low interest rates with periods of both high and low volatility. Again these are features which are difficult to capture without introducing more random factors into a model. This issue is especially important for two types of problem in insurance: the pricing and hedging of long-dated insurance contracts with interest-rate guarantees; and asset-liability modelling and long-term risk-management.

(3) we need more complex models to deal effectively with derivative contracts which are more complex than, say, standard European call options. For example, any contract which makes reference to more than one interest rate should allow these rates to be less than perfectly correlated.

(b) (i)

$$B\left(t,T\right) = \exp\left[-\int_t^T f\left(t,u\right) du\right].$$

Therefore

$$\begin{aligned} B\left(t,T\right) &= \exp\left[-\int_t^T \left(r(t) - \alpha\left(u-t\right)^4\right) du\right] \\ &= \exp\left[-r(t)(T-t) + \frac{\alpha}{5}\left(T-t\right)^5\right] \end{aligned}$$

(ii)

$$R(t,T) = \frac{-1}{T-t} \log B(t,T) \quad \text{if } t < T$$

and therefore

$$\begin{aligned} R(t,T) &= \frac{-1}{T-t}\left[-r(t)(T-t) + \frac{\alpha}{5}\left(T-t\right)^5\right] \\ &= r(t) - \frac{\alpha}{5}\left(T-t\right)^4. \end{aligned}$$

(c) $r(t) = 0.2$, $\alpha = 0.01$ and $T - t = 2$. Then:

$$B(t, T) = \exp\left[-0.2 \times 2 + \frac{0.01}{5} \times 2^5\right]$$
$$= 0.7146.$$