Master in Actuarial Science

Models in Finance

02-02-2015

Time allowed: Two and a half hours (150 minutes)

Instructions:

1. This paper contains 6 questions and comprises 4 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all 6 questions.
6. Begin your answer to each of the 6 questions on a new page.
7. Marks are shown in brackets. Total marks: 200.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider that the share price of a non-dividend paying security is given by

$$S_t = S_0 \exp\left\{h\left(t\right) - \delta B_t\right\}$$

where B_t is a standard Brownian motion, δ is a constant and $h(t)$ is a deterministic function of class $C^2\left(\mathbb{R}\right)$. Assume that the continuously compounded risk-free interest rate r is constant.

   (a) Deduce the stochastic differential equation (SDE) satisfied by the discounted price process $\widetilde{S}_t$. (15)

   (b) Deduce from (a) what should be the function $h(t)$ such that the discounted price $\widetilde{S}_t$ is a martingale and for that $h(t)$ calculate $\mathbb{E}\left[S_t\right]$. (13)

2. Consider the Wilkie model. The force of inflation at time t (measured in years) is given by

$$I\left(t\right) = QMU + QA\left[I\left(t-1\right) - QMU\right] + QSD.QZ(t),$$

where $QMU = 0.03$, $QA = 0.5$, $QSD = 0.005$ and $QZ(t) \sim N(0,1)$.

   (a) What is the the stationary distribution of $I(t)$ in the long run (when $t \to +\infty$). In particular, calculate the expected value of $I(t)$ and the variance of $I(t)$ in the long run (when $t \to \infty$). (15)

   (b) In the Wilkie model, the real yield can be modelled by the equation

$$\begin{aligned}\ln\left(R\left(t\right)\right) = {} & \ln\left(RMU\right) + RA\left[\ln\left(R\left(t-1\right)\right) - \ln\left(RMU\right)\right] \\ & + RBC.CE(t) + RE(t),\end{aligned}$$

   where $CE(t) = CSD.CZ\left(t\right)$, $RE\left(t\right) = RSD.RZ(t)$ and $CZ\left(t\right)$, $RZ(t)$ are series of i.i.d. standard normal random variables. Interpret each term in the equation and the global structure of the equation. (13)

   (c) Explain why the variable modelled for the real yield is $\ln\left(R\left(t\right)\right)$ and not $R\left(t\right)$; and list what are the parameters in the equation that must be estimated from data. (10)

3. Consider European put and call options written on a share with the same strike and maturity.

(a) Explain how and why the strike price, the interest rate and the volatility affect the price of the European call and put options. (13)

(b) Consider that the delta and the gamma of call and put options are known and are given by Δ_c and Γ_c. Deduce formulas for the the delta and gamma of the put option in terms of Δ_c and Γ_c, (i) in the non-dividend case and (ii) in the dividend paying share case. (13)

(c) Write the Black-Scholes PDE (partial differential equation) for the price of a financial derivative using the greeks Δ, Γ and Θ, and considering that we have at time t, $\Delta_c = 0.25$, $\Gamma_c = 0.1$, $\Theta_c = -0.2$, $r = 0.05$, $\sigma = 0.2$ and $S_t = 10$, calculate the price of the call option $c(t, S_t)$. (10)

4. Consider a 2-period recombining binomial model for the non-dividend paying share with price process S_t such that the price at time $t+1$ is either $S_t u$ or $S_t d$ with $u = \frac{1}{d}$. Each period corresponds to one year. Assume that $d = 0.8$ and that $S_0 = 30$€.

   (a) What are the possible values for the continuously compounded risk-free interest rate in order to have an arbitrage free model? Calculate also the risk-neutral probability q if the risk-free interest rate is 10% per year. (10)

   (b) Assuming that the risk-free interest rate is 10% per year, construct the binomial tree and calculate the price of a derivative composed of a sum of an European put option and an European call option. The put option has strike $K_p = 25$ and the call option has a strike $K_c = 40$. The time to maturity of both options is 2 years. (17)

   (c) If the binomial model has N periods, with N large, discuss what are the differences, from the computational point of view, between the recombining binomial model and the non-recombining (general) binomial model (where the up and down factors $u_t(j)$ and $d_t(j)$ depend on the state j and time t). Calculate the number of states in both models when $N = 20$. (12)

5. Consider an investor with a portfolio of N put options and 10000 shares. Assume that the delta of an individual option is -0.25 and that its gamma is 0.1.

(a) Define the greeks delta (Δ), gamma (Γ), lambda (λ) and (ρ) and calculate the number N of put options in the portfolio such that the portfolio has zero delta. (13)

(b) Consider that the investor can invest in two other financial derivatives: the derivative X with price function $F(t, S_t)$ and the derivative Y with price function $G(t, S_t)$. These price functions satisfy

$$\frac{\partial F}{\partial S} = 0.6, \quad \frac{\partial^2 F}{\partial^2 S} = 0.1$$
$$\frac{\partial G}{\partial S} = 0.2, \quad \frac{\partial^2 G}{\partial^2 S} = 0.2$$

Calculate the number of derivatives X and Y that should be added to the portfolio in order to obtain a total portfolio with both delta and gamma equal to zero. (17)

6. Consider the zero-coupon bond market.

(a) Present the stochastic differential equations (SDE), under the risk neutral measure Q, for the short rate in the Vasicek model and in the CIR model, discussing the main differences. (13)

(b) Assume that under the risk-neutral measure $\mathbb{Q}$, the dynamics for the instantaneous forward rate process is

$$df(t,T) = a(t,T)dt + \sigma\left(t.T\right)dW_t,$$

and the dynamics for the zero coupon bond price is

$$dB\left(t.T\right) = B\left(t,T\right)\left[m(t,T)dt + S(t,T)dW_t\right],$$

where

$$m(t,T) = r(t) - \int_t^T a(t,u)du + \left(\int_t^T \sigma(t,u)du\right)^2,$$
$$S\left(t,T\right) = -\int_t^T \sigma(t,u)du.$$

Prove that if the bond market is complete, then

$$\int_t^T a(t,u)du = \left(\int_t^T \sigma(t,u)du\right)^2$$

and $m(t,T) = r(t)$. (16)