Master in Actuarial Science

Models in Finance
02-02-2015

Solutions

1. (a) The discounted price is $\widetilde{S}_t = e^{-rt}S_t = e^{-rt}S_0 \exp\left(h(t) - \delta B_t\right)$. By Itô's lemma (or Itô's formula) applied to $f(t,x) = e^{-rt}\exp(h(t) - \delta B_t)$ (it is a $C^{1,2}$ function):

$$\begin{aligned} d\widetilde{S}_t &= \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)\left(dB_t\right)^2 \\ &= \left(-r\widetilde{S}_t + h'(t)\widetilde{S}_t\right)dt - \delta\widetilde{S}_t dB_t + \frac{1}{2}\delta^2\widetilde{S}_t dt \\ &= \left(h'(t) + \frac{1}{2}\delta^2 - r\right)\widetilde{S}_t dt - \delta\widetilde{S}_t dB_t. \end{aligned}$$

where we have used $\left(dB_t\right)^2 = dt$. Therefore

$$d\widetilde{S}_t = \left(h'(t) + \frac{1}{2}\delta^2 - r\right)\widetilde{S}_t dt - \delta\widetilde{S}_t dB_t.$$

(b) The discounted price process $\widetilde{S}_t$ is a martingale if and only if the drift coefficient in the SDE is zero, that is, $h'(t) + \frac{1}{2}\delta^2 - r = 0$. Indeed, if this occurs then

$$\widetilde{S}_t = \widetilde{S}_0 - \delta\int_0^t \widetilde{S}_u dB_u,$$

and the stochastic integral is a martingale. The function h that satisfies $h'(t) + \frac{1}{2}\delta^2 - r = 0$ is

$$h(t) = \left(r - \frac{1}{2}\delta^2\right)t + C,$$

but at time 0, we have $S_0 = S_0\exp\left\{h\left(0\right) - \delta B_0\right\} = S_0\exp\left\{C\right\}$ and therefore $C = 0$ and

$$h(t) = \left(r - \frac{1}{2}\delta^2\right)t.$$

Moreover, if $\widetilde{S}_t$ is a martingale, then

$$\mathbb{E}\left[\widetilde{S}_t\right] = \widetilde{S}_0 = S_0$$

and therefore

$$\mathbb{E}\left[S_t\right] = \mathbb{E}\left[e^{rt}\widetilde{S}_t\right] = e^{rt}\mathbb{E}\left[\widetilde{S}_t\right] = e^{rt}S_0.$$

2. (a) When $t \to +\infty$, considering the stationary distribution, one must have $\mathbb{E}\left[I\left(t-1\right)\right] = \mathbb{E}\left[I\left(t\right)\right]$ and $Var\left[I\left(t-1\right)\right] = Var\left[I\left(t\right)\right]$. Therefore,

$$\mathbb{E}\left[I\left(t\right)\right] = QMU + QA\mathbb{E}\left[I\left(t\right)\right] - QA.QMU + QSD.\mathbb{E}\left[QZ(t)\right].$$

Hence

$$(1-QA)\mathbb{E}\left[I\left(t\right)\right] = QMU\left(1-QA\right) + 0,$$

and we obtain

$$\mathbb{E}\left[I\left(t\right)\right] = QMU = 0.03.$$

For the variance,

$$Var\left[I\left(t\right)\right] = \left(QA\right)^2 Var\left[I\left(t\right)\right] + \left(QSD\right)^2 Var\left[QZ(t)\right].$$

Hence

$$Var\left[I\left(t\right)\right] = \frac{\left(QSD\right)^2}{1-\left(QA\right)^2} = \frac{\left(0.005\right)^2}{1-\left(0.5\right)^2} = 3.33\times 10^{-5}.$$

The stationary distribution in the long run is the Gaussian distribution $N\left(0.03; 3.33\times 10^{-5}\right)$.

(b) Global structure of the equation: this year's value = long run mean ($\ln(RMU)$)+RA(last year's value − long run mean) + a stochastic shock to the system + another stochastic shock to the system.

The parameter RA is the autoregressive parameter (for the mean-reverting effect). The term $CE(t)$ is a stochastic shock from another process.

The term $CZ(t)$ is the random error term used to model conventional bond yields and $CZ(t)$ and $RZ(t)$ are not combined into a simple series of i.i.d. standard normal random variables because of the correlations that exist between conventional and index-linked bonds.

(c) The real yield of an index-linked bond $R(t)$ is positive, that is the reason why we model $\ln(R(t))$ and not $R(t)$ directly. The parameters to be estimated from data are: RMU, RA, RBC, CSD and RSD.

3. (a) Strike:

Call option: a higher strike price means a lower intrinsic value. A lower intrinsic value means a lower premium.

Put option: a higher strike price will mean a higher intrinsic value and a higher premium.

Interest rates:

Call option: an increase in the risk-free rate of interest will result in a higher value for the option because the money saved by purchasing the option rather than the underlying share can be invested at this higher rate of interest, thus increasing the value of the option.

Put option: higher interest means a lower value (put options can be purchased as a way of deferring the sale of a share: the money is tied up for longer)

Volatility:

The higher the volatility of the underlying share, the greater the chance that the underlying share price can move significantly in favour of the holder of the option before expiry. So the value of an option will increase with the volatility of the underlying share.

(b) The put-call parity:

$$c_t + Ke^{-r(T-t)} = p_t + S_t.$$

Now, if we take the partial derivative with respect to S we obtain:

$$\Delta_c = \Delta_p + 1.$$

If we take the second partial derivative with respect to S, we obtain

$$\Gamma_c = \Gamma_p.$$

In the divided case, the put-call parity is

$$c_t + Ke^{-r(T-t)} = p_t + S_t e^{-q(T-t)}.$$

If we take the derivative with respect to S, we have

$$\Delta_c = \Delta_p + e^{-q(T-t)}.$$

If we take the second derivative with respect to S, we obtain

$$\Gamma_c = \Gamma_p.$$

(c) We know that $\Theta = \frac{\partial f}{\partial t}$, $\Delta = \frac{\partial f}{\partial S}$ and $\Gamma = \frac{\partial^2 f}{\partial S^2}$. Therefore, the Black-Scholes PDE can be transformed in

$$\Theta + rs\Delta + \frac{1}{2}\sigma^2 s^2 \Gamma = rf.$$

Using the numerical data and this equation, we can calculate

$$c(t, S_t) = \frac{-0.2 + 0.05 \times 10 \times 0.25 + \frac{1}{2} \times 0.2^2 \times 10^2 \times 0.1}{0.05} = 2.5.$$

4. (a) $d = 0.8$; $u = \frac{1}{0.8} = 1.25$. In order to obtain an arbitrage free model, we must have $d < e^r < u$. Therefore

$$\ln(0.8) < r < \ln(1.25).$$

Or

$$-0.2231 < r < 0.2231.$$

Since $r > 0$, we must have

$$r \in \left]0, 0.2231\right[.$$

The risk-neutral probability for an up-movement is

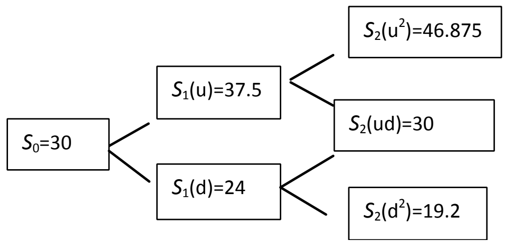
$$q = \frac{e^r - d}{u - d} = \frac{e^{0.10} - 0.8}{1.25 - 0.8} = 0.6782.$$

(b) Binomial tree:

Payoff function of the derivative (call + put):

$$Payoff = \begin{cases} 25 - S_T & \text{if } S_T < 25 \\ 0 & \text{if } 25 \leq S_T \leq 40 \\ S_T - 40 & \text{if } S_T > 40 \end{cases}.$$

Payoff of the derivative: $C_2\left(u^2\right) = 46.875 - 40 = 6.875$, $C_2\left(ud\right) = 0$, $C_2\left(d^2\right) = 25 - 19.2 = 5.8$



Using the usual backward procedure with $r = 0.1$ and $q = 0.6782$:
At time 1: $C_1(u) = \exp(-r)\left[qC_2\left(u^2\right) + (1-q)C_2(ud)\right] = 4.2189$,
$C_1(d) = \exp(-r)\left[qC_2\left(ud^2\right) + (1-q)C_2\left(d^2\right)\right] = 1.6888$
At time 0, the price is $C_0 = \exp(-r)\left[qC_1(u) + (1-q)C_1(d)\right] = 3.0807$.

(c) the non-recombinig binomial model allows for different values of volatility when in different states (it allows different up and down factors for different states): $u_t(j)$ and $d_t(j)$ vary with t and j. Therefore, the number of states at time N is 2^N states: if N is large, it is a big number with exponential growth (for computational purposes), since computation times even for simple derivative securities are at best proportional to the number of states. For example, with 20 periods, at time $t = 20$ we have $2^{20} = 1048600$ states.

In the recombining binomial model, it is assumed that the volatility is the same at all states (the up and down factors are the same irrespective of wether they appear in the binomial tree). At time N we have $N+1$ possible states (linear growth with N) instead of 2^N. For example, in a 20-period model, we have 21 states at time $t = 20$, instead of 1048600 states. Therefore, with this model the computing times are substantially reduced.

5. (a) Let $f(t,s)$ be the value at time t of a derivative when the price of the the underlying asset at t is $S_t = s$.

Delta of the derivative and vega:

$$\Delta = \frac{\partial f}{\partial s},$$
$$\Gamma = \frac{\partial^2 f}{\partial s^2},$$
$$\rho = \frac{\partial f}{\partial r},$$
$$\lambda = \frac{\partial f}{\partial q},$$

where r is the risk free interest rate and q is the dividend yield on the underlying asset.

In order to have a porfolio with zero delta, $\Delta_p \times N + \Delta_S \times$number of shares$= 0$. Since $\Delta_p = -0.25$ and $\Delta_S = 1$, we have

$$N = \frac{10000}{0.25} = 40000.$$

(b) We have $\Delta_X = 0.6$, $\Delta_Y = 0.2$, $\Gamma_X = 0.1$, $\Gamma_Y = 0.2$. Let N_X be the number of derivatives X and N_Y be the number of derivatives Y in the portfolio. In order to have a zero delta and a zero gamma portfolio:

$$\begin{cases} 0.6N_X + 0.2N_y = 0 \\ 40000 \times 0.1 + 0.1N_X + 0.2N_Y = 0 \end{cases}$$

It is easy to solve this linear system os 2 equations. The solution is

$$\begin{cases} N_X = 8000, \\ N_Y = -24000. \end{cases}$$

6. (a) The Vasicek model has the dynamics, under the risk-neutral measure Q:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma d\widetilde{W}(t)$$

where $\widetilde{W}$ is a standard Brownian motion under Q.

The Cox-Ingersoll-Ross (CIR) model has the dynamics under Q:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\widetilde{W}(t).$$

The critical difference between the two models occurs in the volatility, which is increasing in line with the square-root of $r(t)$

for the CIR model. Since this diminishes to zero as $r(t)$ approaches zero, and provided σ^2 is not too large ($r(t)$ will never hit zero provided $\sigma^2 \leq 2\alpha\mu$), we can guarantee that $r(t)$ will not hit zero. Consequently all other interest rates will also remain strictly positive.

(b) If the bond market is complete then the discounted zero-coupon bond price $\widetilde{B}(t,T) = \exp\left(-\int_0^t r(s)\,ds\right) B(t,T)$ is a martingale with respect to the risk-neutral probability measure $\mathbb{Q}$. By the Itô formula applied to $f(t,x) = \exp\left(-\int_0^t r(s)\,ds\right) x$, and by the fundamental theorem of integral calculus, we have that

$$\begin{aligned} d\widetilde{B}(t,T) &= -r(t)\exp\left(-\int_0^t r(s)\,ds\right) B(t,T)\,dt + \exp\left(-\int_0^t r(s)\,ds\right) dB(t,T) \\ &= -r(t)\widetilde{B}(t,T)\,dt + \widetilde{B}(t,T)\left[m(t,T)dt + S(t,T)dW_t\right] \\ &= \widetilde{B}(t,T)\left[(m(t,T) - r(t))\,dt + S(t,T)dW_t\right]. \end{aligned}$$

In order to be a martingale, the drift coefficient must be zero, that is, $m(t,T) - r(t) = 0$. Therefore

$$\int_t^T a(t,u)du = \left(\int_t^T \sigma(t,u)du\right)^2.$$