Microeconomics

Joana Pais

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Properties of the expenditure function

Theorem 1.7: Properties of the Expenditure Function:

If u(·) is continuous and strictly increasing, then $e(p, u)$ is:

- Zero when u takes on the lowest level of utility in $\mathcal{U}$.
- Continuous on its domain $\mathbb{R}^n_{++} \times u$.
- For all $p >> 0$, strictly increasing and unbounded above in u.
- Increasing in p.
- Homogeneous of degree 1 in p.
- Concave in p.

  If, in addition, $u(\cdot)$ is strictly quasiconcave, we have:
- Shephard's lemma: $e(p, u)$ is differentiable in p at (p^0, u^0) with $p^0 >> 0$, and $\partial e(p^0, u^0)/\partial p^0 = x_i^h(p^0, u0)$, $i = 1, ..., n$.

Relations between UMP and EMP

Theorem 1.8: Relations between indirect utility and expenditure functions

Let $v(p, y)$ and $e(p, u)$ be the indirect utility function and expenditure function for some consumer whose utility function is continuous and strictly increasing. Then for all $p >> 0$, $y \geq 0$, and $u \in \mathcal{U}$:

- $e(p, v(p, y)) = y$.
- $v(p, e(p, u)) = u$.

Theorem 1.9: Duality between Marshallian and Hicksian demand functions

Under Assumption 1.2 we have the following relations between the Hicksian and Marshallian demand functions for $p >> 0$, $y \geq 0$, $u \in \mathcal{U}$, and $i = 1, ..., n$:

- $x_i(p, y) = x_i^h(p, v(p, y))$.
- $x_i^h(p, u) = x_i(p, e(p, u))$.

Relative prices and real income

Theorem 1.10: Homogeneity and budget balancedness

Under Assumption 1.2 the consumer demand function $x_i(p, y)$, $i = 1, ..., n$, is homogeneous of degree zero in all prices and income, and it satisfies budget balancedness, $p.x(p, y) = y$ for all (p, y).

Income and substitution effects

Theorem 1.11: The Slutsky equation

Let $x(p, y)$ be the consumer's Marshallian demand system. Let u^* be the level of utility the consumer achieves at prices p and income y. Then,

$$\underbrace{\frac{\partial x_i(p, y)}{\partial p_j}}_{\text{TE}} = \underbrace{\frac{\partial x_i^h(p, u^*)}{\partial p_j}}_{\text{SE}} - \underbrace{x_j(p, y).\frac{\partial x_i(p, y)}{\partial y}}_{\text{IE}}$$

Theorem 1.12: Negative own-substitution terms

Let $x_i^h(p, u)$ be the Hicksian demand for good i. Then,

$$\frac{\partial x_i^h(p,u)}{\partial p_i} = 0,\ i = 1, \ldots, n.$$

Theorem 1.13: The law of demand

A decrease in the own price of a normal good will cause quantity demanded to increase. If an own price decrease causes a decrease in quantity demanded, the good must be inferior.

Income and substitution effects

Theorem 1.14: Symmetric substitution terms

Let $x^h(p, u)$ be the consumer's system of Hicksian demands and suppose that $e(\cdot)$ is twice continuously differentiable. Then,

$$\frac{\partial x_i^h(p,u)}{\partial p_j} = \frac{\partial x_j^h(p,u)}{\partial p_i},\ i, j = 1, \ldots, n.$$

Theorem 1.15: Negative Semidefinite substitution matrix

Let $x^h(p, u)$ be the consumer's system of Hicksian demands and let

$$\sigma(p,u) = \begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \cdots & \frac{\partial x_1^h(p,u)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^h(p,u)}{\partial p_1} & \cdots & \frac{\partial x_n^h(p,u)}{\partial p_n} \end{bmatrix}$$

be called the **substitution matrix**, containing all the Hicksian substitution terms. Then, the matrix $\sigma(p, u)$ is negative semidefinite.

Income and substitution effects

Theorem 1.15: Negative Semidefinite substitution matrix

Let $x(p, y)$ be the consumer's Marshallian demand system. Define the *ij*th Slutsky term as

$$\frac{\partial x_i(p,y)}{\partial p_j} + x_j(p, y).\frac{\partial x_i(p,y)}{\partial y}$$

and form the entire $n \times n$ **Slutsky matrix** of price and income responses as follows:

$$s(p, y) = \begin{bmatrix} \frac{\partial x_1(p,y)}{\partial p_1} + x_1(p, y).\frac{\partial x_1(p,y)}{\partial y} & \dots & \frac{\partial x_1(p,y)}{\partial p_n} + x_n(p, y).\frac{\partial x_1(p,y)}{\partial y} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(p,y)}{\partial p_1} + x_1(p, y).\frac{\partial x_n(p,y)}{\partial y} & \dots & \frac{\partial x_n(p,y)}{\partial p_n} + x_n(p, y).\frac{\partial x_n(p,y)}{\partial y} \end{bmatrix}$$

Some elasticity relations

Definition 1.6: Demand elasticities and income shares

Let $x_i(p, y)$ be the consumer's Marshallian demand for good i. Then let

$$\eta_i \equiv \frac{\partial x_i(p,y)}{\partial y} \frac{y}{x_i(p,y)}$$
$$\epsilon_{ij} \equiv \frac{\partial x_i(p,y)}{\partial p_j} \frac{p_j}{x_i(p,y)}$$

and let

$$s_i \equiv \frac{p_i x_i(p,y)}{y} \text{ so that } s_i \geq 0 \text{ and } \sum_{i=1}^{n} s_i = 1.$$

Some elasticity relations

Theorem 1.17: Aggregation in consumer demand

Let $x(p, y)$ be the consumer's Marshallian demand system. Let η_i, ϵ_i, and s_i, for $i, j = 1, \ldots, n$ be as defined before. Then the following relations must hold among income shares, price, and income elasticities of demand:

- Engel aggregation: $\sum_{i=1}^{n} s_i \eta_i = 1$.
- Cournot aggregation$\sum_{i=1}^{n} s_i \epsilon_{ij} = -s_j$, $j = 1, \ldots, n$

Chapter 2:
Topics in consumer theory

Expenditure and consumer preferences

Consider any function of prices and utility, $E(p, u)$, that may or may not be an expenditure function. Now suppose that E satisfies the expenditure function properties 1 to 7 of Theorem 1.7, so that it is continuous, strictly increasing and unbounded above in u, as well as increasing, homogeneous of degree one, concave, and differentiable in p. Thus E 'looks like' an expenditure function. We shall show that E must then be an expenditure function. Specifically, we shall show that there must exist a utility function on $\mathbb{R}^n_+$ whose expenditure function is precisely E. Indeed, we shall give and explicit procedure for constructing this utility function.

Expenditure and consumer preferences

To see how the construction works, choose $(p^0, u^0) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$, and evaluate E there to obtain the number to construct the (closed) "half - space" in the consumption set,

$$A(p^0, u^0) \equiv \{x \in \mathbb{R}^n_+ : p^0.x \geq E(p^0, u^0)\}.$$

Notice that $A(p^0, u^0)$ is closed convex set containing all points on and above the hyperplane, $p^0 \cdot x = E(p^0, u^0)$. Now choose different prices p^1, keep u^0 fixed, and construct the closed convex set,

$$A(p^1, u^0) \equiv \{x \in \mathbb{R}^n_+ : p^1.x \geq E(p^1, u^0)\}.$$

Expenditure and consumer preferences

Imagine proceeding like this for all prices $p >> 0$ and forming the infinite intersection,

$$A(u^0) \equiv \cap_{p>>0} A(p, u^0) = \{x \in \mathbb{R}^n_+ : p.x \geq E(p, u^0) \text{ for all } p >> 0\}.$$

Theorem 2.1: Constructing a utility function from an expenditure function

Let $E : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy properties 1 through 7 of the expenditure function (Theorem 1.7). Let $A(u)$ be as defined in the last slide. Then the function $u : \mathbb{R}^n_+ \times \mathbb{R}_+$ given by

$$u(x) \equiv max\{u \geq 0 : x \in A(u)\}$$

is increasing, unbounded above, and quasiconcave.

Theorem 2.2: The expenditure function of derived utility, u, is E

Let $E(p, u)$ defined on $\mathbb{R}^n_{++} \times \mathbb{R}^n_{+}$ satisfy properties 1 to 7 of the expenditure function (Theorem 1.7) and let $u(x)$ be derived from E as in Theorem 2.1. Then, for all non-negative prices and utility,

$$E(p, u) = \min_x p.x \text{ s.t. } u(x) \geq u.$$

That is, $E(p, u)$ is the expenditure function generated by derived utility $u(x)$.

Theorem 2.3: Duality between direct and indirect utility

Suppose that $u(x)$ is quasiconcave and differentiable on $\mathbb{R}^n_{++}$ with strictly positive partial derivatives there. Then for all $x \in \mathbb{R}^n_{++}$, $v(p, p.x)$, the indirect utility function generated by $u(x)$, achieves a minimum in p on $\mathbb{R}^n_{++}$, and

$$u(x) = \min_{p \in \mathbb{R}^n_{++}} v(p, p.x).$$

Theorem 2.4: Duality and the system of inverse demands

Let $u(x)$ be the consumer's utility function. Then, the inverse demand function for good i associated with income $y = 1$ is given by:

$$p_i(x) = \frac{\partial u(x)/\partial x_i}{\sum_{j=1}^{n} x_j(\partial u(x)/\partial x_j)}.$$

Integrability

Theorem 2.5: Budget balancedness and symmetry imply homogeneity

If $x(p, y)$ satisfies budget balancedness and its Slutsky matrix is symmetric, then it is homogeneous of degree 0 in p and y.

Theorem 2.6: Integrability theorem

A continuously differentiable function $x : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^{n}_{+}$ is the demand function generated by some increasing, quasiconcave utility function if (and only if when utility is continuous, strictly increasing, and strictly quasiconcave) it satisfies budget balancedness, symmetry, and negative semidefiniteness.

Revealed preference

Definition 2.1: Weak axiom of revealed preference (WARP)

A consumer's choice behaviour satisfies WARP if for every distinct pair of bundles x^0, x^1 with x^0 chosen at prices p^0 and x^1 chosen at prices p^1,

$$p^0 \cdot x^1 \leq p^0 \cdot x^0 \Rightarrow p^1 \cdot x^0 > p^1 \cdot x1.$$

In other words, WARP holds if whenever x^0 is revealed preferred to x^1, x^1 is never revealed preferred to x^0.

Uncertainty

Until now, we have assumed that decision makers act in a world on absolute certainty. The consumer knows the prices of all commodities and knows that any feasible consumption bundle can be obtained with certainty. Clearly, economic agents in the real world cannot always operate under such pleasant conditions. Many economic decisions contain some element of uncertainty, namely uncertainty about the outcome of the choice that is made. Whereas the decision maker may know the probabilities of different possible outcomes, the final result of the decision cannot be known until it occurs.

Definition 2.2: Simple gambles

Let $A = \{a_1, ..., a_n\}$ be the set of outcomes. Then $\mathcal{G}_S$, the set of simple gambles (on A), is given by

$$\mathcal{G}_S \equiv \{(p_1 \circ a_1, \ldots, p_n \circ a_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}.$$

Preferences

Axiom 1: Completeness. For any two gambles, g and g' in $\mathcal{G}$, either $g \succsim g'$ or $g' \succsim g$.

Axiom 2: Transitivity. For any three gambles, g, g', and g'' in $\mathcal{G}$, if $g \succsim g'$ and $g' \succsim g''$, then $g \succsim g''$.

Axiom 3: Continuity. For any gamble g in $\mathcal{G}$, there is some probability, $\alpha \in [0, 1]$ such that $g \sim (\alpha \circ a_1, (1 - \alpha) \circ a_n)$.

Axiom 4: Monotonicity. For all probabilities $\alpha, \beta \in [0, 1]$,

$$(\alpha \circ a_1, (1 - \alpha) \circ a_n) \succsim (\beta \circ a_1, (1 - \beta) \circ a_n)$$

if and only if $\alpha \geq \beta$.

Axiom 5: Substitution. If $g = (p_1 \circ g^1, \ldots, p_k \circ g^k)$ and $h = (p_1 \circ h^1, \ldots, p_k \circ h^k)$ are in $\mathcal{G}$, and if $h^i \sim g^i$ for every i, then $h \sim g$.

Axiom 6: Reduction to Simple Gambles. For any gamble $g \in \mathcal{G}$, if $(p_1 \circ a_1, \ldots, p_n \circ a_n)$ is the simple gamble induced by g, then $(p_1 \circ a_1, \ldots, p_n \circ a_n) \sim g$.

Definition 2.3: Expected utility property

The utility function $u : \mathcal{G} \rightarrow \mathbb{R}$ has the expected utility property if, for every $g \in \mathcal{G}$,

$$u(g) = \sum_{i=1}^{n} p_i u(a_i).$$

where $(p_1 \circ a_1, \ldots, p_n \circ a_n)$ is the simple gamble induced by g.

Von-Neumann Morgenstern utility

Theorem 2.7: Existence of a VNM utility function on $\mathcal{G}$

Let preferences over gambles in $\mathcal{G}$ satisfy axioms G1 to G6. Then, there exists a utility funtion $u : \mathcal{G} \to \mathbb{R}$ representing $\succsim$ on $\mathcal{G}$, such that u has the expected utility property.

Theorem 2.8: VNM utility functions are unique up to affine transformations

Suppose that the VNM utility funtion $u(\cdot)$ represents $\succsim$. Then the VNM utility function $v(\cdot)$ represents those same preferences if and only if for some scalar α and some scalar $\beta > 0$,

$$v(g) = \alpha + \beta u(g)$$

for all gambles g.

Definition 2.4: Risk aversion, risk neutrality, and risk loving

Let $u(\cdot)$ be an individual's VNM utility function for gambles over non-negative levels of wealth. Then for the simple gamble $g = (p_1 \circ w_1, ..., p_n \circ w_n)$, the individual is said to be

- risk averse at g if $u(E(g)) > u(g)$;
- risk neutral at g if $u(E(g)) = u(g)$;
- risk loving at g if $u(E(g)) < u(g)$.

If for every non-degenerate simple gamble g, the individual is, for example, risk averse at g, then the individual is said imply to be risk averse (or risk averse on $\mathcal{G}$ for emphasis). Similarly, an individual can be defined to be risk neutral and risk loving (on $\mathcal{G}$).

Definition 2.5: Certainty equivalent and risk premium

The certainty equivalent of any simple gamble g over wealth levels is an amount of wealth, CE, offered with certainty, such that $u(g) \equiv u(CE)$. The risk premium is an amount of wealth P such that $u(g) \equiv u(E(g) - P)$. Clearly, $P \equiv E(g) - CE$.

Definition 2.6: The Arrow-Pratt measure of absolute risk aversion

The Arrow-Pratt measure of absolute risk aversion is given by

$$R_a(w) \equiv \frac{-u''(w)}{u'(w)}.$$