Models in Finance - Class 3

Master in Actuarial Science

João Guerra

ISEG

Stochastic integrals

- Motivation : Consider a "differential equation" with "noise" of type:

$$\frac{dX}{dt} = b(t, X_t) + \sigma(t, X_t) \text{"} \frac{dB_t}{dt} \text{"}.$$

- "$\frac{dB_t}{dt}$" is stochastic "noise". Does not exist in classical sense since B is not differentiable.
- "Stochastic differential equation" (SDE) in integral form :

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \text{"} \int_0^t \sigma(s, X_s) dB_s \text{"} \tag{1}$$

- How to define the integral:

$$\int_0^T u_s \mathrm{d}B_s \quad ? \tag{2}$$

where B is a Brownian motion and u is an appropriate adapted process.

- Note: the SDE's that we deal with are the continuous time versions of the equations used to define time series (processes in discrete time). Example: a zero-mean random walk can be defined by:

$$X_t = X_{t-1} + \sigma Z_t,$$

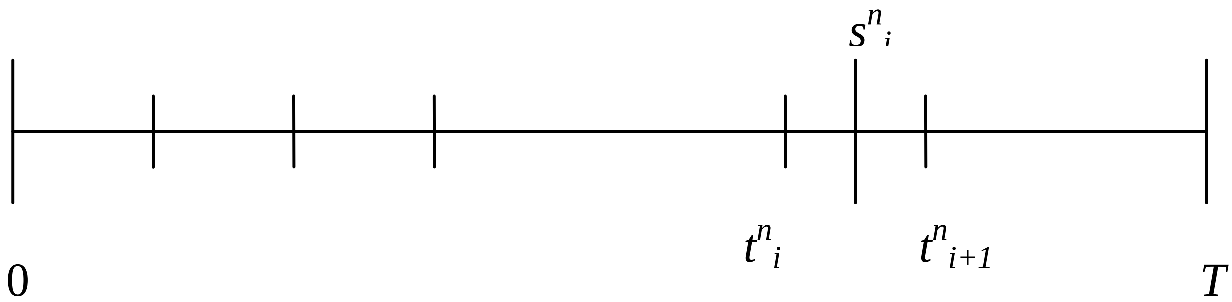
where Z_t is a standard normal r.v. (the Z_i variables are called white noise). This equation is a stochastic difference equation and is equivalent to $\Delta X_t = \sigma Z_t$. Its solution is $X_t = X_0 + \sigma \sum_{s=1}^{t} Z_s$.

- In continuous time, the analog of a zero-mean random walk is a zero-mean Brownian motion W_t.

- First strategy: Consider the integral (2)
- Consider a sequence of partitions of $[0, T]$ and a sequence of points:

$$\tau_n: \ 0 = t_0^n < t_1^n < t_2^n < \cdots < t_{k(n)}^n = T$$
$$s_n: \ t_i^n \leq s_i^n \leq t_{i+1}^n, \quad i = 0, \ldots, k(n) - 1,$$

such that $\lim_{n\to\infty} \sup_i \left(t_{i+1}^n - t_i^n\right) = 0$.



Riemann-Stieltjes (R-S) integral:

$$\int_0^T f dg := \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(s_i^n\right) \Delta g_i,$$

where $\Delta g_i := g\left(t_{i+1}^n\right) - g\left(t_i^n\right)$, if the limit exists and is independent of the sequences τ_n and s_n.

- If g is a differentiable function and f is continuous the (R-S) integral is well defined: $\int_0^T f(t)\, dg(t) = \int_0^T f(t)\, g'(t)\, dt$.
- In the Bm case B, it is clear that $B'(t)$ does not exist, so we cannot define the path integral:

$$\int_0^T u_t(\omega)\, dB_t(\omega) \overset{\times}{\neq} \int_0^T u_t(\omega)\, B_t'(\omega)\, dt$$

- Problem: The integral $\int_0^T B_t(\omega)\, dB_t(\omega)$ does not exist as a R-S integral. How to define the integral (2)?

- We will construct the stochastic integral $\int_0^T u_t dB_t$ using a probabilistic approach.

Definition

Consider processes u of class $L^2_{a,T}$, which is defined as the class of processes $u = \{u_t,\ t \in [0, T]\}$, such that:

1. u is adapted and measurable.
2. $E\left[\int_0^T u_t^2 dt\right] < \infty$.

- Condition 1. allows us to show that $\int_0^t u_s ds$ is $\mathcal{F}_t$-measurable
- Condition 2. allows us to show that u as a map of two variables t and ω belongs to the space $L^2([0, T] \times \Omega)$ and that:

$$E\left[\int_0^T u_t^2 dt\right] = \int_0^T E\left[u_t^2\right] dt.$$

- idea: we will define $\int_0^T u_t dB_t$ for $u \in L^2_{a,T}$ as a limit in mean-square (i.e., a limit in $L^2(\Omega)$) of integrals of simple processes.

Stochastic Itô integral for simple processes

Definition

$u \in \mathcal{S}$ (set of simple processes in $[0, T]$) is called a simple process if

$$u_t = \sum_{j=1}^{n} \phi_j \mathbf{1}_{(t_{j-1}, t_j]}(t), \qquad (3)$$

where $0 = t_0 < t_1 < \cdots < t_n = T$, and the r.v. ϕ_j are square-integrables ($E\left[\phi_j^2\right] < \infty$) and $\mathcal{F}_{t_{j-1}}$-measurable

Definition

If u is a simple process of form (3) ($u \in \mathcal{S}$) then the stochastic Itô integral of u with respect to Bm B is:

$$\int_0^T u_t dB_t := \sum_{j=1}^{n} \phi_j \left(B_{t_j} - B_{t_{j-1}}\right).$$

Example

Consider the simple process

$$u_t = \sum_{j=1}^{n} B_{t_{j-1}} \mathbf{1}_{(t_{j-1}, t_j]}(t).$$

Then

$$\int_0^T u_t dB_t = \sum_{j=1}^{n} B_{t_{j-1}} \left(B_{t_j} - B_{t_{j-1}}\right).$$

Then (why?)

$$E\left[\int_0^T u_t dB_t\right] = \sum_{j=1}^{n} E\left[B_{t_{j-1}} \left(B_{t_j} - B_{t_{j-1}}\right)\right]$$

$$= \sum_{j=1}^{n} E\left[B_{t_{j-1}}\right] E\left[B_{t_j} - B_{t_{j-1}}\right] = 0.$$

Proposition: (Isometry property or norm preservation property). Let $u \in \mathcal{S}$. Then:

$$E\left[\left(\int_0^T u_t dB_t\right)^2\right] = E\left[\int_0^T u_t^2 dt\right] = \int_0^T E\left[u_t^2\right] dt. \tag{4}$$

Proof.

With $\Delta B_j := B_{t_j} - B_{t_{j-1}}$, we have (Exercise (homework): justify all the steps in this proof):

$$\begin{aligned} E\left[\left(\int_0^T u_t dB_t\right)^2\right] &= E\left[\left(\sum_{j=1}^n \phi_j \Delta B_j\right)^2\right] \\ &= \sum_{j=1}^n E\left[\phi_j^2 \left(\Delta B_j\right)^2\right] + 2\sum_{i<j}^n E\left[\phi_i \phi_j \Delta B_i \Delta B_j\right]. \end{aligned}$$

□

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Proof.

(cont.) Note that since $\phi_i \phi_j \Delta B_i$ is $\mathcal{F}_{j-1}$-measurable and ΔB_j is indepedent of $\mathcal{F}_{j-1,}$ then

$$\sum_{i<j}^n E\left[\phi_i \phi_j \Delta B_i \Delta B_j\right] = \sum_{i<j}^n E\left[\phi_i \phi_j \Delta B_i\right] E\left[\Delta B_j\right] = 0.$$

On the other hand, since ϕ_j^2 is $\mathcal{F}_{j-1}$-measurable and ΔB_j is independent of $\mathcal{F}_{j-1,}$

$$\begin{aligned} \sum_{j=1}^n E\left[\phi_j^2 \left(\Delta B_j\right)^2\right] &= \sum_{j=1}^n E\left[\phi_j^2\right] E\left[\left(\Delta B_j\right)^2\right] \\ &= \sum_{j=1}^n E\left[\phi_j^2\right] \left(t_j - t_{j-1}\right) = \\ &= E\left[\int_0^T u_t^2 dt\right]. \end{aligned}$$

□

- Other properties of $\int_0^T u_t dB_t$ for $u \in \mathcal{S}$:
  1. Linearity: If $u,v \in \mathcal{S}$:
  $$\int_0^T (au_t + bv_t)\, dB_t = a\int_0^T u_t dB_t + b\int_0^T v_t dB_t. \tag{5}$$
  2. Zero mean:
  $$E\left[\int_0^T u_t dB_t\right] = 0. \tag{6}$$

Exercíse: Prove the property 2.
Exercise: Compute $\int_0^5 f(s)\, dB_s$ with $f(s) = 1$ if $0 \leq s \leq 2$ and $f(s) = 4$ if $2 < s \leq 5$ and what is the distribution of the resulting r.v.?

Itô integral

Lemma

If $u \in L^2_{a,T}$ then exists a sequence of simple processes $\left\{u^{(n)}\right\}$ such that
$$\lim_{n\to\infty} E\left[\int_0^T \left|u_t - u_t^{(n)}\right|^2 dt\right] = 0. \tag{7}$$

Proof: see Oksendal's book or Nualart lecture notes:
http://www.math.ku.edu/~nualart/StochasticCalculus.pdf

Definition

The Itô stochastic integral of $u \in L^2_{a,T}$ is defined as the limit (in the $L^2(\Omega)$ sense):

$$\int_0^T u_t dB_t = \lim_{n\to\infty} (L^2) \int_0^T u_t^{(n)} dB_t, \tag{8}$$

where $\left\{u^{(n)}\right\}$ is a sequence of simple processes satisfying (7).

Properties of the Itô integral

- Properties of the Itô integral $\int_0^T u_t dB_t$ for $u \in L^2_{a,T}$.
  1. Isometry (or norm preservation):
  $$E\left[\left(\int_0^T u_t dB_t\right)^2\right] = E\left[\int_0^T u_t^2 dt\right] = \int_0^T E\left[u_t^2\right] dt. \tag{9}$$
  2. Zero mean:
  $$E\left[\int_0^T u_t dB_t\right] = 0 \tag{10}$$
  3. Linearity:
  $$\int_0^T (au_t + bv_t)\, dB_t = a\int_0^T u_t dB_t + b\int_0^T v_t dB_t. \tag{11}$$
  4. The process $\left\{\int_0^t u_s dB_s, t \geq 0\right\}$ is a martingale.
  5. The sample paths of $\left\{\int_0^t u_s dB_s, t \geq 0\right\}$ are continuous.

Example

Let us show that

$$\int_0^T B_t dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}T.$$

Since $u_t = B_t$, let us consider the sequence of simple processes

$$u_t^n = \sum_{j=1}^{n} B_{t_{j-1}^n} \mathbf{1}_{\left(t_{j-1}^n, t_j^n\right]}(t),$$

with $t_j^n := \frac{j}{n}T$.

Example

(cont.)

$$\begin{aligned}
\int_0^T B_t dB_t &= \lim_{n\to\infty}(L^2) \int_0^T u_t^{(n)} dB_t = \\
&= \lim_{n\to\infty}(L^2) \sum_{j=1}^{n} B_{t_{j-1}^n}\left(B_{t_j^n} - B_{t_{j-1}^n}\right) \\
&= \lim_{n\to\infty}(L^2) \frac{1}{2}\sum_{j=1}^{n}\left[\left(B_{t_j^n}^2 - B_{t_{j-1}^n}^2\right) - \left(B_{t_j^n} - B_{t_{j-1}^n}\right)^2\right] \\
&= \frac{1}{2}\left(B_T^2 - T\right),
\end{aligned}$$

where we used: $E\left[\left(\sum_{j=1}^n \left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] = 0$ and
$\frac{1}{2}\sum_{j=1}^n \left(B_{t_j^n}^2 - B_{t_{j-1}^n}^2\right) = \frac{1}{2}B_T^2$.

- Let us prove that $E\left[\left(\sum_{j=1}^{n}\left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] = 0$. Using the independence of increments and $E\left[\left(\Delta B_{t_j^n}\right)^2\right] = \Delta t_j^n$, then

$$
\begin{aligned}
E\left[\left(\sum_{j=1}^{n}\left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] &= E\left[\left(\sum_{j=1}^{n}\left[\left(\Delta B_{t_j^n}\right)^2 - \Delta t_j^n\right]\right)^2\right] \\
&= \sum_{j=1}^{n} E\left[\left(\Delta B_{t_j^n}\right)^2 - \Delta t_j^n\right]^2 .
\end{aligned}
$$

Using the fact that $E\left[\left(B_t - B_s\right)^{2k}\right] = \frac{(2k)!}{2^k \cdot k!}\left(t-s\right)^k$, then

$$
\begin{aligned}
E\left[\left(\sum_{j=1}^{n}\left(\Delta B_{t_j^n}\right)^2 - T\right)^2\right] &= \sum_{j=1}^{n}\left[3\left(\Delta t_j^n\right)^2 - 2\left(\Delta t_j^n\right)^2 + \left(\Delta t_j^n\right)^2\right] \\
&= 2\sum_{j=1}^{n}\left(\Delta t_j^n\right)^2 = 2T\sup_j\left|\Delta t_j^n\right| \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

- Note: By formula $E\left[\left(B_t - B_s\right)^{2k}\right] = \frac{(2k)!}{2^k \cdot k!}\left(t-s\right)^k$ we have that

$$
\begin{aligned}
Var\left[\left(\Delta B\right)^2\right] &= E\left[\left(\Delta B\right)^4\right] - \left(E\left[\left(\Delta B\right)^2\right]\right)^2 \\
&= 3\left(\Delta t\right)^2 - \left(\Delta t\right)^2 = 2\left(\Delta t\right)^2 .
\end{aligned}
$$

We also know that

$$E\left[\left(\Delta B\right)^2\right] = \Delta t.$$

Therefore, if Δt is small, the variance of $\left(\Delta B\right)^2$ is very small when compared with its expected value $\Longrightarrow$therefore when $\Delta t \to 0$ or "$\Delta t = dt$", we have:

$$\left(dB_t\right)^2 \approx dt. \tag{12}$$