Lévy processes and applications - Theoretical overview

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Lévy-Khintchine formula

- $(\Omega, \mathcal{F}, P)$.
- A Lévy Process $X = (X(t), t \geq 0)$ is essentially a stochastic process with stationary and indepedent increments.
- Key-formula: the Lévy-Khintchine formula:

$$E\left[e^{iuX(t)}\right] = e^{t\eta(u)}$$

where

$$\eta(u) = ibu - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}(x)\right] \nu(dx),$$

$b, \sigma \in \mathbb{R}$ and ν is a Lévy measure on $\mathbb{R} - \{0\}$ such that, for all $u \in \mathbb{R}$:

$$\int_{\mathbb{R}-\{0\}} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty.$$

Lévy-Khintchine formula

- Let $\sigma = \nu = 0$. Then
$$E\left[e^{iuX(t)}\right] = e^{itub}$$
and $X(t) = bt$ is a deterministic motion in a stright line (b is the velocity of the motion - drift)
- Let $\sigma \neq 0$ and $\nu = 0$. Then
$$E\left[e^{iuX(t)}\right] = \exp\left[t\left(ibu - \frac{1}{2}u^2\sigma^2\right)\right],$$
which is the characteristic of a Gaussian r.v. with mean tb and variance σ^2.In fact $X(t)$ is a Brownian motion with drift.

Theoretical Overview

Lévy-Khintchine formula

- Let ν be a finite measure ($\lambda = \int_{\mathbb{R}} \nu(dx) < \infty$). Then
$$\eta(u) = ib'u - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[e^{iux} - 1\right]\nu(dx),$$
with $b' = b - \int_{0<|x|<1} x\nu(dx)$.
- If $\nu = \lambda\delta_h$ with $\lambda > 0$ then
$$X(t) = b't + \sigma B(t) + N(t),$$
where $N(t)$ is a Poisson process of intensity λ with jumps of size $|h|$. Note that $E\left[e^{iuN(t)}\right] = \exp\left[\lambda t\left(e^{iuh} - 1\right)\right]$

Lévy-Khintchine formula

- Let $\nu = \sum_{i=1}^{m} \lambda_i \delta_{h_i}$. Then

$$X(t) = b' t + \sigma B(t) + N_1(t) + \cdots N_m(t),$$

where the $N_i' s$ are independent Poisson processes (also independent of B), with N_i of intensity λ_i with jumps of size $|h_i|$.

- General case with finite ν corresponds to jump sizes in a continuum of possibilities (continuum of Poisson processes - jumps of arbitrary size).

Remarks

- Most subtle case: infinite measure case: $\int_{0<|x|<1} |x| \, \nu(dx) = \infty$ and $\int_{0<|x|<1} |x|^2 \, \nu(dx) < \infty$.
- Then $e^{iux} - 1$ may no longer be ν-integrable, but $e^{iux} - 1 - iux\mathbf{1}_{\{0<|x|<1\}}(x)$ is allways ν-integrable
- When ν is finite we can write:

$$X(t) = bt + \sigma B(t) + \sum_{0 \leq s \leq t} \Delta X(s),$$

- $\Delta X(s)$ is the jump at time s.

Lévy-Itô decomposition

- For each Borel set $A \in \mathbb{R} - \{0\}$, let $N(t, A) = \# \{0 \leq s \leq t : \Delta X(s) \in A\}$
- Fix t and A: then $N(t, A)$ is a r.v.
- Fix $\omega \in \Omega$ and t: then $N(t, \cdot)(\omega)$ is a measure
- Fix A. Then $\{N(t, A), t \geq 0\}$ is a Poisson process with intensity $\nu(A)$ with

$$\sum_{0 \leq s \leq t} \Delta X(s) = \int_{\mathbb{R} - \{0\}} x N(t, dx)$$

- In the case of infinite measure ν, we have the Lévy-Itô decomposition:

$$X(t) = bt + \sigma B(t) + \int_{0<|x|<1} x \left[N(t, dx) - t\nu(dx)\right] + \\ + \int_{|x| \geq 1} x N(t, dx)$$

Applebaum, D. (2004). Lévy Processes and Stochastic Caculus. Cambridge University Press. - (Overview)

Papantaloleon, A. An Introduction to Lévy Processes with Applications in Finance. arXiv:0804.0482v2.