Microeconomics - Chapter 3

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Chapter 3:
Theory of the firm

Primitive notions

- A firm is an entity created by individuals for some purpose. This entity will typically acquire inputs and combine them to produce output. Inputs are purchased on input markets and these expenditures are the firm's costs. Output is sold on product markets and the firm earns revenue from these sales.
- From an empirical point of view, assuming firms maximise profits leads to predictions of firm behaviour which are time and again borne out by the evidence.

Production

Production is the process of transforming inputs into outputs. The fundamental reality firms must contend with in the process is technological feasibility. The state of technology determines and restricts what is possible in combining inputs to produce output, and there are several ways we can represent this constraint. The most general way is to think of the firm as having a **production possibility set**, $Y \subset R^m$, where each vector $y = (y_1, \ldots, y_m) \in Y$ is a **production plan** whose components indicate the amounts of the various inputs and outputs. A common convention is to write elements of $y \in Y$ so that $y_i < 0$ if resource i is used up in the production plan, and $y_i > 0$ if resource i is produced in the production plan.

Production

The production possibility set is by far the most general way to characterise the firm's technology because it allows for multiple inputs and multiple outputs. Often, however, we will want to consider firms producing only a single product from many inputs. For that, it is more convenient to describe the firm's technology in terms of a **production function**.

When there is only one output produced by many inputs, we shall denote the amount of output by y, and the amount of input i by x_i, so that with n inputs, the entire vector of inputs is denoted by $x = (x_1, \ldots, x_n)$. Of course, the input vector as well as the amount of output must non-negative, so we require $x \geq 0$, and $y \geq 0$.

Production

A production function simply describes for each vector of inputs the amount of output that can be produced. The production function, f , is therefore a mapping from $\mathbb{R}_+^n$ into $\mathbb{R}_+$. When we write $y = f(x)$, we mean that y unit of output (and no more) can be produced using the input vector x. We shall maintain the following assumption on the production function f .

Production

Assumption 3.1: Properties of the production function

The production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$, and $f(0) = 0$.

Production

When the production function is differentiable, its partial derivative $\partial f(x)/\partial x_i$ is called **marginal product** of input i and gives the rate at which output changes per additional unit of input i employed. If f is strictly increasing and everywhere continuously differentiable, then $\partial f(x)/\partial x_i > 0$ for 'almost all" input vectors. We often assume for simplicity that the strict inequality holds.

For any fixed level of output y the set of input vectors producing y units of output is called the y-level **isoquant**. An isoquant is then just a level set of f . We shall denote this set by $Q(y)$. That is,

$$Q(y) \equiv \{x \geq 0 : f(x) = y\}$$

For an input vector x, the isoquant through x is the set of input vector producing the same output as x, namely, $Q(f(x))$.

Production

The **marginal rate of technical substitution (*MRTS*)** measures the rate at which one input can be substituted for another without changing the amount of output produced. Formally, the marginal rate of technical substitution of input j for input i when the current input vector is x, denoted $MRTS_{ij}(x)$, is denoted as the ratio of marginal products,

$$MRTS_{ij}(x) = \frac{\partial f(x)/\partial x_i}{\partial f(x)/\partial x_j}.$$

In the two-input case, the $MRTS_{12}(x^1)$ is the absolute value of the slope of the isoquant through x^1 at the point x^1.

Production

Definition 3.1: Separable utility functions
Let $N = \{1, \ldots, n\}$ index the set of all inputs, and suppose that these inputs can be partitioned into $S > 1$ mutually exclusive and exhaustive subsets $N_1, \ldots, N_S$. The production function is called weakly separable if the *MRTS* between two inputs within the same group is independent of inputs used in other groups:

$$\frac{\partial(f_i(x)/\partial f_j(x))}{\partial x_k} = 0 \text{ for all } i, j \in N_S \text{ and } k \notin N_S,$$

where f_i and f_j are the marginal products of inputs i and j. When $S > 2$, the production function is called strongly separable if the *MRTS* between two inputs from any two groups, namely from the same group, is independent of all inputs outside those two groups:

$$\frac{\partial(f_i(x)/\partial f_j(x))}{\partial x_k} = 0 \text{ for all } i \in N_S,\ j \in N_t, \text{ and } k \notin N_S \cup N_t.$$

Production

Definition 3.2: Elasticity of substitution

Given a production function $f(x)$, the elasticity of substitution of input j for input i at the point $x^0 \in \mathbb{R}^n_{++}$ is defined as:

$$\sigma_{ij}(x^0) \equiv \left(\frac{d \ln MRTS_{ij}(x(r))}{d \ln r} \Big|_{r = x_j^0 / x_i^0} \right)^{-1},$$

where $x(r)$ is the unique vector of inputs $x = (x_1, \ldots, x_n)$ such that (i) $x_j / x_i = r$, (ii) $x_k = x_k^0$ for $k \neq i, j$, and (iii) $f(x) = f(x^0)$.

Production

Theorem 3.1: (Shephard) Homogeneous production functions are concave

Let $f(x)$ be a production function satisfying Assumption 3.1 and suppose it is homogeneous of degree $\alpha \in (0, 1]$. Then $f(x)$ is a concave function of x .

Returns to scale and varying proportions

We frequently want to know how output responds as the amounts of different inputs are varied. For instance, in the **short run**, the period of time in which at least one input is fixed, output can be varied only by changing the amounts of some inputs but not others. As amounts of the variable inputs are changed, the proportions in which fixed and variable inputs are used are also changed. 'Returns of variable proportions' refer to how output response in this situation. In the **long run**, the firm is free to vary all inputs, and classifying production functions by their 'returns to scale' is one way of describing how output responds in this situation. Specially, returns to scale refer to how responds when all inputs are varied in the same proportion, i.e., when the entire 'scale' of operation is increased or decreased proportionally.

Returns to scale and varying proportions

Elementary measures of returns to varying proportions include the **marginal product**, $MP_i(x) \equiv f_i(x)$, and the **average product**, $AP_i(x) \equiv f(x)/x_i$, of each product.
The **output elasticity of input** i, measuring the percentage response of output to a 1 per cent change in input i, is given by $\mu_i(x) \equiv f_i(x)x_i/f(x) = MP_i(x)/AP_i/(x)$. Each of these is a local measure, defined at a point. The scale properties of the technology may be defined either locally or globally. A production function is said to have globally constant, increasing or decreasing returns to scale according to the following definitions.

Returns to scale and varying proportions

Definition 3.3: Global returns to scale

A production function $f(x)$ has the property of (globally):

- Constant returns of scale if $f(tx) = tf(x)$ for all $t > 0$ and all x.
- Increasing returns of scale if $f(tx) > tf(x)$ for all $t > 1$ and all x.
- Decreasing returns of scale if $f(tx) < tf(x)$ for all $t > 1$ and all x.

Returns to scale and varying proportions

Definition 3.4: Local returns to scale

The elasticity of scale at the point x is defined as:

$$\mu(x) \equiv lim_{t \to 1} \frac{d\ln[f(tx)]}{d\ln(t)} = \frac{\sum_{i=1}^{n} f_i(x)x_i}{f(x)}.$$

Returns to scale are locally constant, increasing, or decreasing as $\mu(x)$ is equal to, greater than, or less than one. The elasticity of scale and the output elasticities of the inputs are related as follows:

$$\mu(x) = \sum_{i=1}^{n} \mu_i(x).$$

Cost

The firm's cost of output is precisely the expenditure it must make to acquire the inputs used to produce that output. In general, the technology will permit every level of output to be produced by a variety of input vectors, and all such possibilities can be summarised by the level sets of the production cost. Then firm must decide, therefore, which of the possible production plans it will use. If the object of the firm is to maximise profits, it will necessarily choose the least costly, or cost-minimising, production plan for every level of output.

Cost

We will assume throughout that firms are perfectly competitive on their input markets and therefore they face fixed input prices. Let $w = (w_1, \ldots, w_n) \geq 0$ be a vector of prevailing market prices at which the firm can buy inputs $x = (x_1, \ldots, x_n)$. Because the firm is a profit maximiser, it will choose to produce some level of output while using that input vector requiring the smallest money outlay. One can speak therefore of 'the' cost of output y – it will be the cost at prices w of the least costly vector of input capable of producing y.

Cost

Definition 3.5: The cost function

The cost function, defined for all input prices $w \gg 0$ and all input levels $y \in f(\mathbb{R}^n_+)$ is the minimum-value function,

$$c(w, y) \equiv min_{x \in \mathbb{R}^n_+} w \cdot x \text{ s.t. } f(x) \geq y.$$

If $x(w, y)$ solves the cost-minimisation problem, then

$$c(w, y) = w \cdot x(w, y).$$

Cost

Theorem 3.2: Properties of the cost function

If f is continuous and strictly increasing, then $c(w, y)$ is:

- Zero when $y = o$.
- Continuous on its domain.
- For all $w \gg 0$, strictly increasing and unbounded above in y.
- Increasing in w.
- Homogeneous of degree one in w.
- Concave in w.

  Moreover if f is strictly quasiconcave:
- Shephard's lemma: $c(w, y)$ is differentiable in w at (w^0, y^0) whenever $w^0 \gg 0$, and

$$\frac{\partial c(w^0, y^0)}{\partial w_i} = x_i(w^0, y^0),\ i = 1, \ldots, n.$$

Cost

Theorem 3.3: Properties of conditional input demands

Suppose the production function satisfies Assumption 3.1 and the associated cost function is twice continuously differentiable. Then:

- $x(w, y)$ is homogeneous of degree zero in w.
- The substitution matrix, defined and denoted

$$\sigma^*(w, y) \equiv \begin{bmatrix} \frac{\partial x_1(w,y)}{\partial w_1} & \cdots & \frac{\partial x_1(w,y)}{\partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n(w,y)}{\partial w_1} & \cdots & \frac{\partial x_n(w,y)}{\partial w_n} \end{bmatrix}$$

is symmetric and negative semidefinite. In particular, the negative semidefiniteness property implies that $\frac{\partial x_i(w,y)}{\partial w_i} \leq 0$ for all i.

Cost

Theorem 3.4: Cost and conditional input demands when production is homothetic

Suppose the production function satisfies Assumption 3.1 and the associated cost function is twice continuously differentiable. Then:

- When the production function is homothetic,
  - the cost function is multiplicatively separable in input prices and output and can be written $c(w, y) = h(y)c(w, 1)$, where $h(y) > 0$ and $c(w, 1)$ is the unit cost function, or the cost of 1 unit of output;
  - the conditional input demands are multiplicatively separable in input prices and can be written $x(w, y) = h(y)x(w, 1)$, where $h'(y) > 0$ and $x(w, 1)$ is the conditional input demand for 1 unit of output.

Cost

- When the production function is homogeneous of degree $\alpha > 0$,
  - $c(w, y) = y^{1/\alpha} c(w, 1)$,
  - $x(w, y) = y^{1/\alpha} x(w, 1)$.

Cost

Definition 3.6: The short-run, or restricted, cost function

Let the production function be $f(z)$, where $z \equiv (x, \overline{x})$. Suppose that x is a subvector of variable inputs and $\overline{x}$ is a subvector of fixed inputs. Let w and $\overline{w}$ be the associated input prices for the variable and fixed inputs, respectively. The short-run, or restricted, total cost function is defined as

$$sc(w, \overline{w}, y; \overline{x}) \equiv min_x w \cdot x + \overline{w} \cdot \overline{x} \text{ s.t. } f(x, \overline{x}) \geq y.$$

If $x(w, \overline{w}, y; \overline{x})$ solves this minimisation problem, then

$$sc(w, \overline{w}, y; \overline{x}) = w \cdot x(w, \overline{w}, y; \overline{x}) + \overline{w} \cdot \overline{x}.$$

The optimised cost of the variable inputs, $w \cdot x(w, \overline{w}, y; \overline{x})$, is called **total variable cost**. The cost of the fixed inputs, $\overline{w} \cdot \overline{x}$, is called **total fixed cost**.

Duality in production

Theorem 3.5: Recovering a production function from a cost function

Let $c : \mathbb{R}^n_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy properties 1 to 7 of a cost function given in Theorem 3.2. Then the function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ defined by

$$f(x) \equiv max\{y \geq 0 : w \cdot x \geq c(w, y), \text{for all } w \gg 0\}.$$

is increasing, unbounded above, and quasiconcave. Moreover, the cost function generated by f is c.

Duality in production

We can also state an integrability-type theorem for input demand: if $x(w, y)$ summarises the conditional input demand behaviour of some firm, under what conditions can we conclude that this behaviour is consistent with the hypothesis that each level of output produced by the firm was produced at minimum cost? As in the case of demand, the answer will depend on being able to recover a cost function that generates the given input demands. That is, those demands will be consistent with cost minimisation at each output level it and only if there is a cost function c satisfying

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w, y),\ i = 1, \ldots, n.$$

Duality in production

Theorem 3.6: Integrability for cost functions

A continuously differentiable function $x(w, y)$ mapping $\mathbb{R}^n_{++} \times \mathbb{R}_+$ into $\mathbb{R}_+$ is the conditional input demand function generated by some strictly increasing, quasiconcave production function if and only if it is homogeneous of degree zero in w, its substitution matrix, whose ijth entry is $\partial x_i(w, y)/\partial w_j$, is symmetric and negative semidefinite, and $w \cdot x(w, y)$ is strictly increasing y .

The profit function

Definition 3.7: The profit function

The firm's profit functions depends only on input and output prices and is defined as the maximum-value function,

$$\pi(p, w) \equiv max_{x,y \geq 0}\ py - w \cdot x \text{ s.t. } f(x) \geq y.$$

The profit function

Theorem 3.7: Properties of the profit function

If f satisfies Assumption 3.1, then for $p \geq 0$ and $w \geq 0$, the profit function $\pi(p, w)$, where well-defined, is continuous and

- Increasing in p.
- Decreasing in w.
- Homogeneous of degree one in (p, w).
- Convex in (p, w).
- Differentiable in $(p, w) \gg 0$. Moreover, under the additional assumption that f is strictly concave (Hotelling's lemma), $\frac{\partial \pi(p,w)}{\partial p} = y(p, w)$, and $\frac{-\partial \pi(p,w)}{\partial w_i} = x_i(p, w)$, $i = 1, \ldots, n$.

The profit function

Theorem 3.8: Properties of output supply and input demand functions

Suppose that f is a strictly concave production function satisfying Assumption 3.1 and that its associated profit function, $\pi(p, y)$, is twice continuously differentiable. Then, for all $p > 0$ and $w \gg 0$ where it is well defined, output supply and input demand functions:

- satisfy homogeneous of degree zero:
  $y(tp, tw) = y(p, w)$ for all $t > 0$
  $x_i(tp, tw) = x_i(p, w)$ for all $t > 0$ and $i = 1, \dots, n$
- have own-price effects as follows:
  $\frac{\partial y(p,w)}{\partial p} \geq 0,$
  $\frac{\partial x_i(p,w)}{\partial w_i} \leq 0$ for all $t > 0$ for all $i = 1, \dots, n$

The profit function

Theorem 3.8: Properties of output supply and input demand functions

- have the substitution matrix:

$$\begin{bmatrix} \frac{\partial y(p,w)}{\partial p} & \frac{\partial y(p,w)}{\partial w_1} & \cdots & \frac{\partial y(p,w)}{\partial w_n} \\ \frac{-\partial x_1(p,w)}{\partial p} & \frac{-\partial x_1(p,w)}{\partial w_1} & \cdots & -\frac{\partial x_1(p,w)}{\partial w_n} \\ \vdots & \ddots & \vdots & \\ \frac{-\partial x_n(p,w)}{\partial p} & \frac{-\partial x_n(p,w)}{\partial w_1} & \cdots & -\frac{\partial x_n(p,w)}{\partial w_n} \end{bmatrix}$$

which is symmetric and positive semidefinite.

The profit function

Theorem 3.9: The short-run, or restricted, profit function

Suppose that $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is strictly concave and satisfies Assumption 3.1. For $k < n$, let $\overline{x} \in \mathbb{R}^k_+$ be a subvector of fixed inputs and consider $f(x, \overline{x})$ as a function of the subvector of variable inputs $x \in \mathbb{R}^{n-k}$. Let w and $\overline{w}$ be the associated input process for variable and fixed inputs, respectively. The short-run, or restricted, profit function is defined as:

$$\pi(p, w, \overline{w}, \overline{x}) \equiv max_{y,x} py - w \cdot x - \overline{w} \cdot \overline{x} \text{ s.t. } f(x, \overline{x}) \geq y.$$

The solutions $y(p, w, \overline{w}, \overline{x})$ and $x(p, w, \overline{w}, \overline{x})$ are called the short-run output supply and variable input demand functions, respectively.

The profit function

For all $p > 0$ and $w \gg 0$, $\pi(p, w, \overline{w}, \overline{x})$ where well-defined, is continuous in p and w, increasing in p, decreasing in w, and convex in (p, w). If $\pi(p, w, \overline{w}, \overline{x})$ is twice continuously differentiable, $y(p, w, \overline{w}, \overline{x})$ and $x(p, w, \overline{w}, \overline{x})$ possess all three properties listed in Theorem 3.8 with respect to output and variable input prices.