Financial Markets and Instruments (Lecture 5)

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- Vamos ver de novo o caso da combinação de dois activos com risco supondo que não são possíveis vendas a descoberto.
- Vamos procurar identificar aquilo que se chama a fronteira das carteiras eficientes.

$$\overline{R}_p = X_c\overline{R}_c + X_s\overline{R}_s = X_c\overline{R}_c + (1 - X_c)\overline{R}_s$$

$$\sigma_p = (X_c^2\sigma_c^2 + X_s^2\sigma_s^2 + 2X_cX_s\rho_{cs}\sigma_c\sigma_s)^{½}$$

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Empresas	Retorno Esperado	Desvio Padrão
Colonel Motors [C]	14%	6%
Separated Edison [S]	8%	3%

Para: $\rho_{cs} = +1$

- $\sigma_p = X_c\sigma_c + \sigma_s - X_c\sigma_c$

- $X_c = \dfrac{\sigma_p - \sigma_s}{\sigma_c - \sigma_s}$

$$\overline{R}_P = \left[\overline{R}_S - \frac{\overline{R}_C - \overline{R}_S}{\sigma_c - \sigma_S}\sigma_S\right] + \left[\frac{\overline{R}_C - \overline{R}_S}{\sigma_c - \sigma_S}\right]\sigma_P$$

X_C	0	0.2	0.4	0.5	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11	11.6	12.8	14.0
σ_P	3.0	3.6	4.2	4.5	4.8	5.4	6.0

Table 5-1 The expected Return and Standard Deviation of a Portfolio of Colonel Motors and Separated Edison When r = +1

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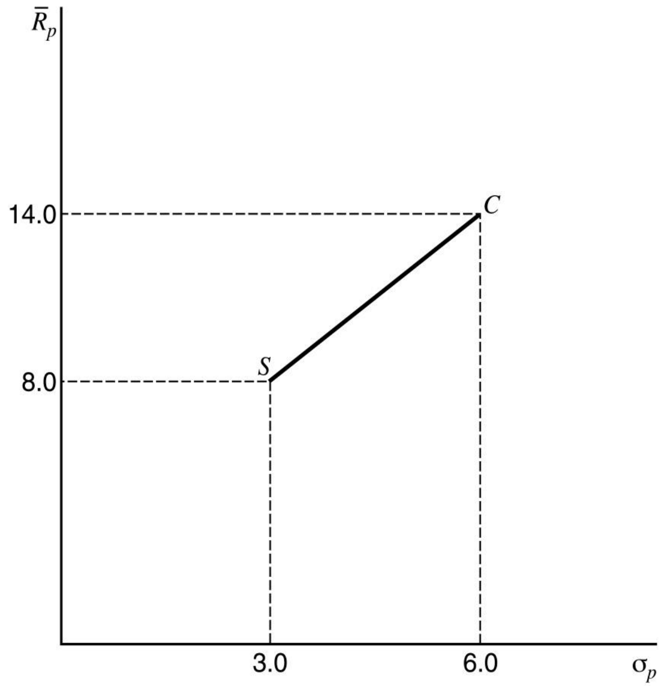


Para: $\rho_{cs} = +1$

Para: $\rho_{es} = -1$

$$\sigma_p = X_c\sigma_c - \sigma_s + X_c\sigma_s \Rightarrow X_c = \frac{\sigma_p + \sigma_s}{\sigma_c + \sigma_s}$$

ou:

$$\sigma_p = \sigma_s - X_c\sigma_S - X_c\sigma_C \Rightarrow X_c = \frac{\sigma_s - \sigma_p}{\sigma_s + \sigma_c}$$

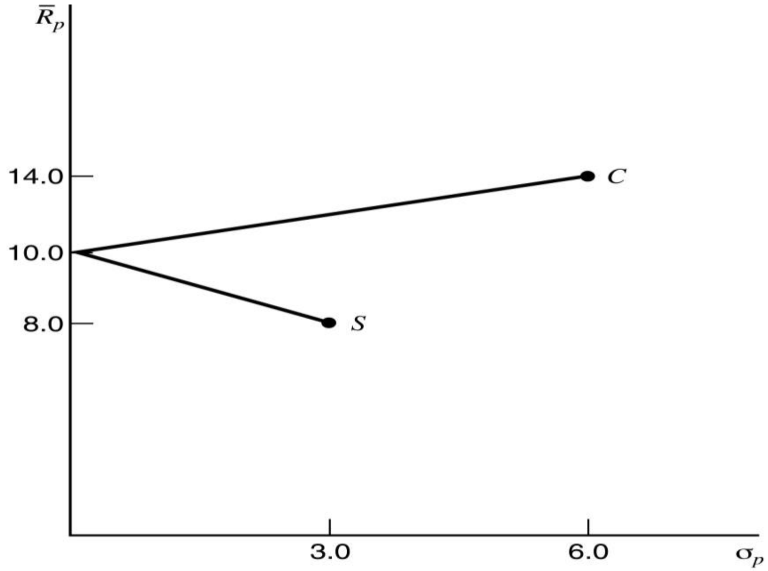
$$\overline{R}_P = \left[\overline{R}_S + \frac{\overline{R}_C - \overline{R}_S}{\sigma_c - \sigma_S}\sigma_S\right] + \left[\frac{\overline{R}_C - \overline{R}_S}{\sigma_c - \sigma_S}\right]\sigma_P$$

ou:

$$\overline{R}_P = \left[\overline{R}_S + \frac{\overline{R}_C - \overline{R}_S}{\sigma_c + \sigma_S}\sigma_S\right] - \left[\frac{\overline{R}_C - \overline{R}_S}{\sigma_c + \sigma_S}\right]\sigma_P$$

X_C	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	3.0	1.2	0.6	2.4	4.2	6.0

Table 5-2 The expected Return and Standard Deviation of a Portfolio of Colonel Motors and Separated Edison When r = –1



Para: $\rho_{cs} = -1$

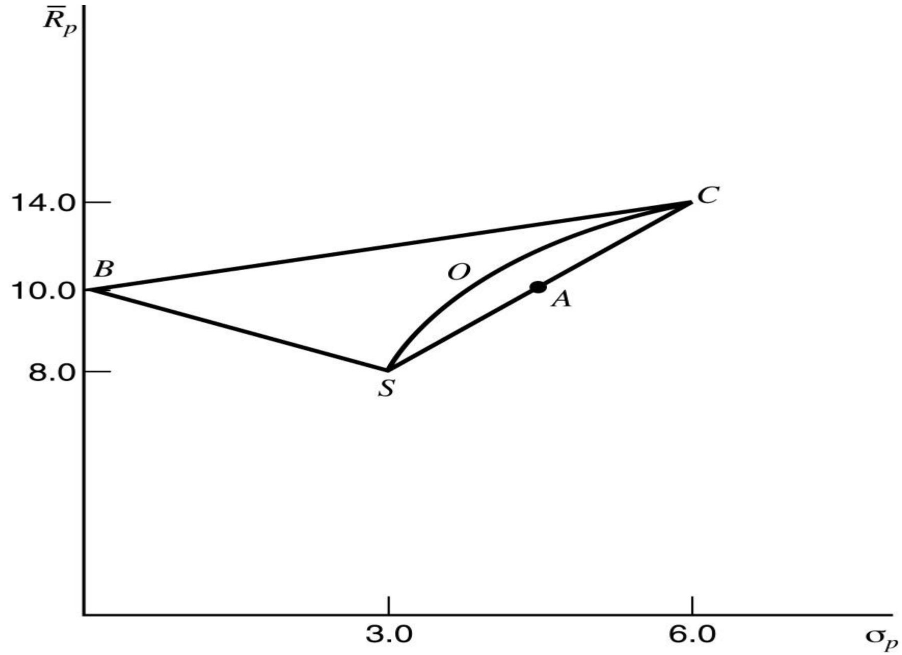


FIGURE 5-3 Relationship between expected return and standard deviation for various correlation coefficients.

Para: $\rho_{cs} = 0$

- $\sigma_p = (X_c^2\sigma_c^2 + X_s^2\sigma_s^2)^{1/2}$

X_C	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	3.00	2.68	3.00	3.79	4.84	6.0

Table 5-3 The Expected Return and Standard Deviation for a Portfolio of Colonel Motors and Separated Edison with r = 0

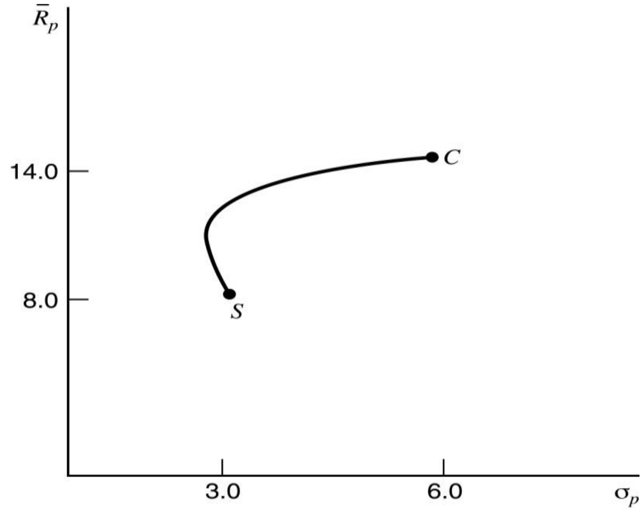


Figure 5-4 Relationship between expected return and standard deviation when r = 0.

X_C	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	3.00	3.17	3.65	4.33	5.13	6.00

Table 5-4 The expected Return and Standard Deviation of a Portfolio of Colonel Motors and Separated Edison When r = 0.5

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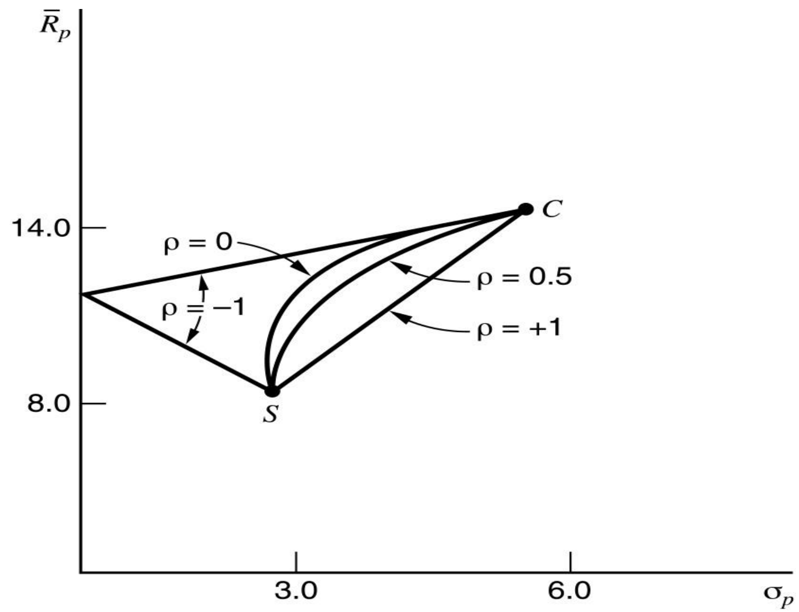


FIGURE 5-5 Relationship between expected return and standard deviation of return for various correlation coefficients.

$$\sigma_p = (X_c^2\sigma_c^2 + X_s^2\sigma_s^2 + 2X_cX_s\rho_{cs}\sigma_c\sigma_s)^{1/2}$$

Sendo

$$X_s = (1 - X_s)$$

Ponto que minimiza o risco da carteira

Ponto que minimiza σ_p (ponto de variância mínima)

$$\frac{\partial \sigma_p}{\partial X_C} = 0 \Leftrightarrow$$

- $$X_c = \frac{\sigma_s^2 - \sigma_c \sigma_s \rho_{cs}}{\sigma_c^2 + \sigma_s^2 - 2\sigma_c \sigma_s \rho_{cs}} = \frac{\sigma_s^2 - \sigma_{cs}}{\sigma_c^2 + \sigma_s^2 - 2\sigma_{cs}}$$

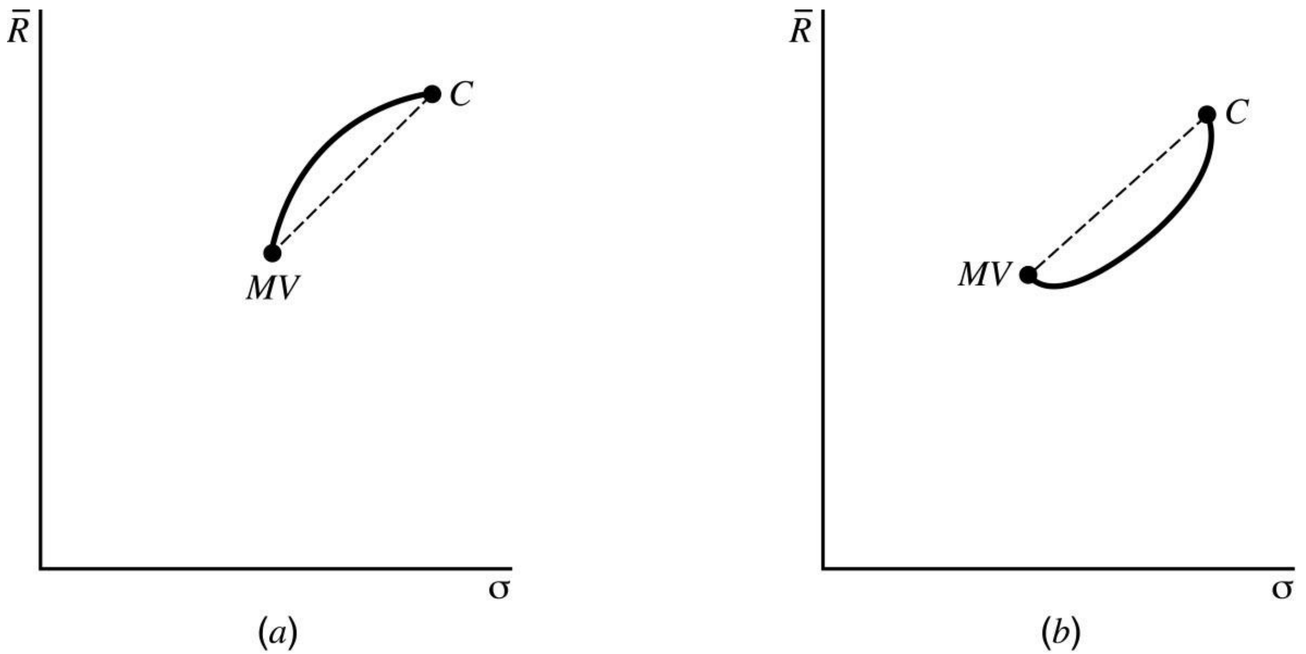


FIGURE 5-6 Various possible relationships for expected return and standard deviation when the minimum variance portfolio and Colonel Motors are combined.

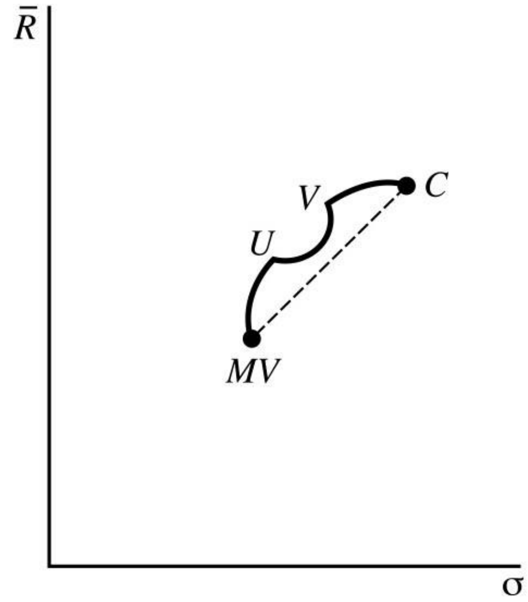


FIGURE 5-6 (continued)

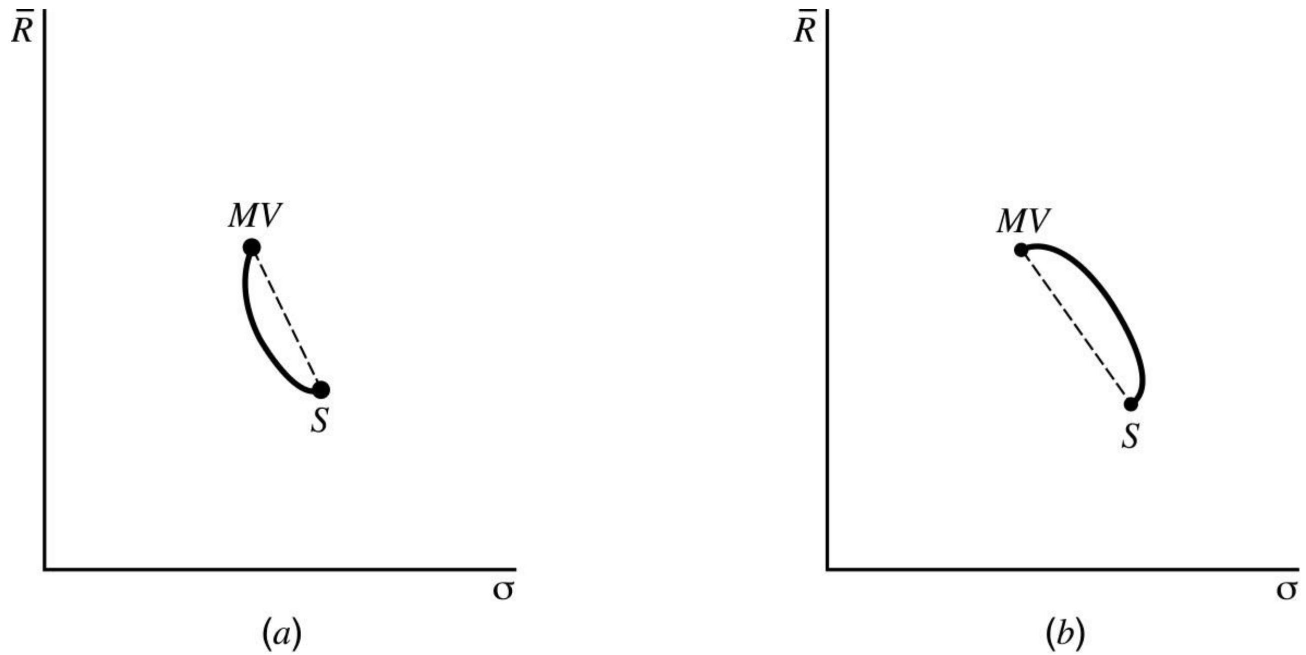
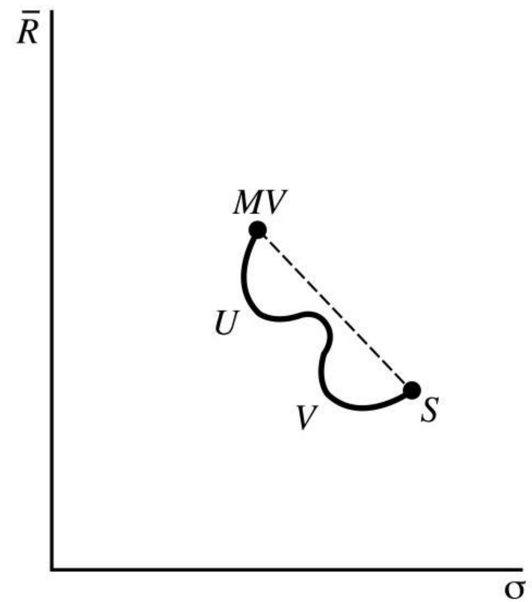


FIGURE 5-7 Various possible relationships between expected return and standard deviation of return when the minimum variance portfolio is combined with portfolio *S*.



(c)

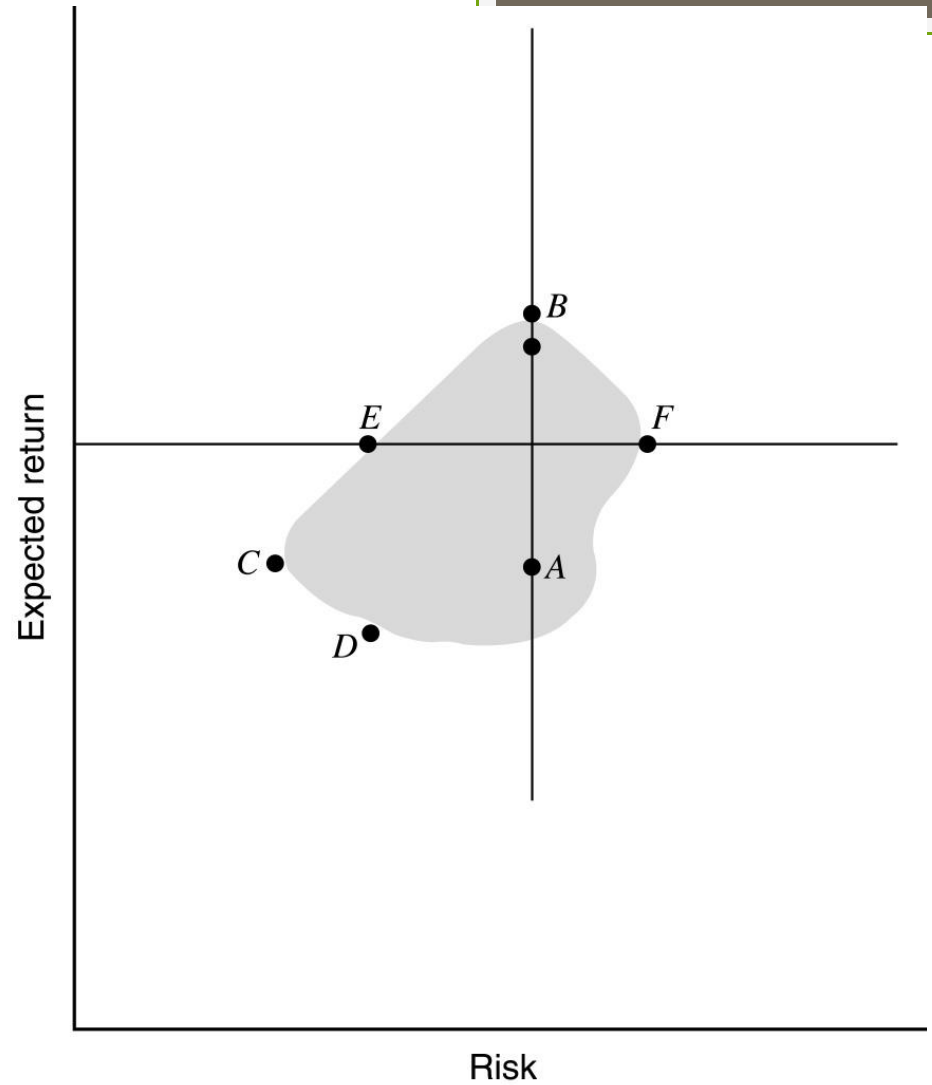


FIGURE 5-8 Risk and return possibilities for various assets and portfolios.

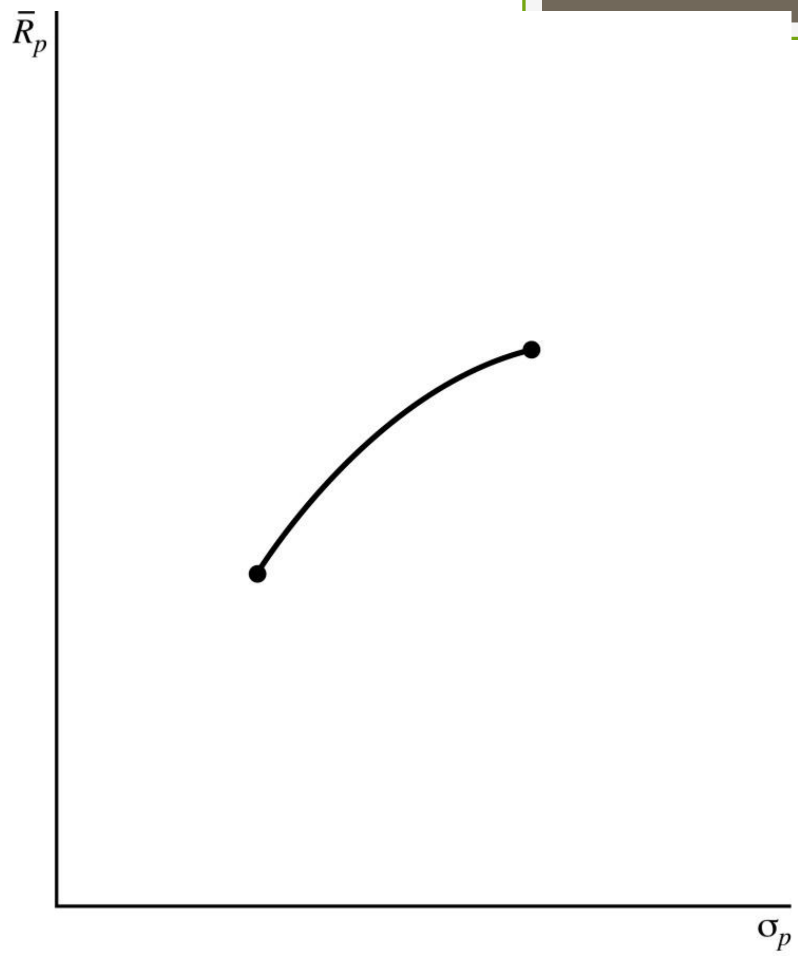


FIGURE 5-9 The efficient frontier.

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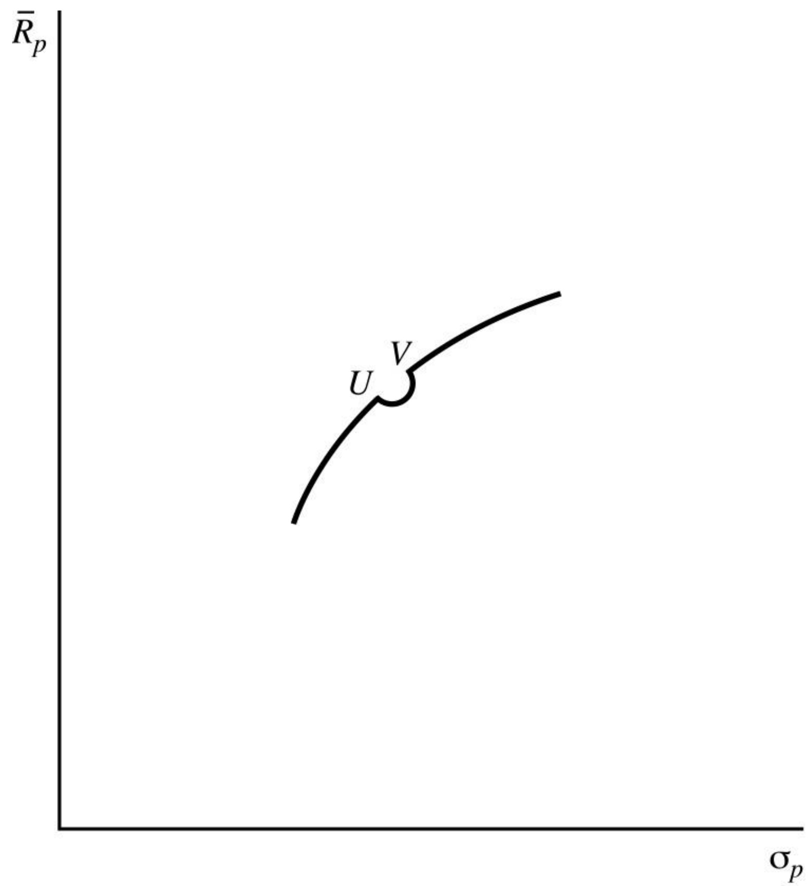

FIGURE 5-10 An impossible shape for the efficient frontier.

- Introdução das vendas a descoberto no nosso exemplo anterior com dois activos.

Table 5-5 The Expected Return and Standard Deviation When Short Sales Are Allowed

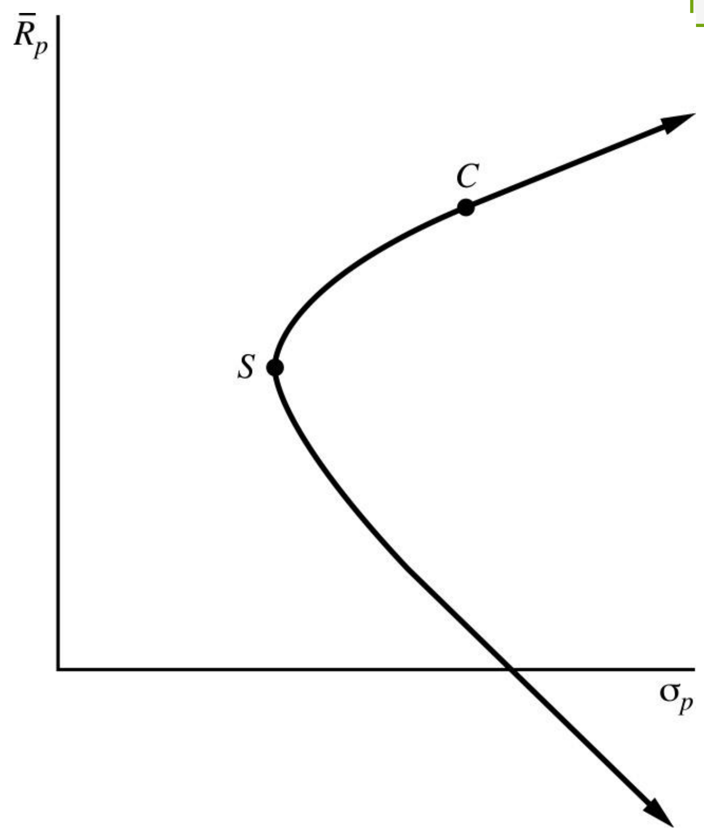

FIGURE 5-11 Expected return standard deviation combinations of Colonel Motors and Separated Edison when short sales are allowed.

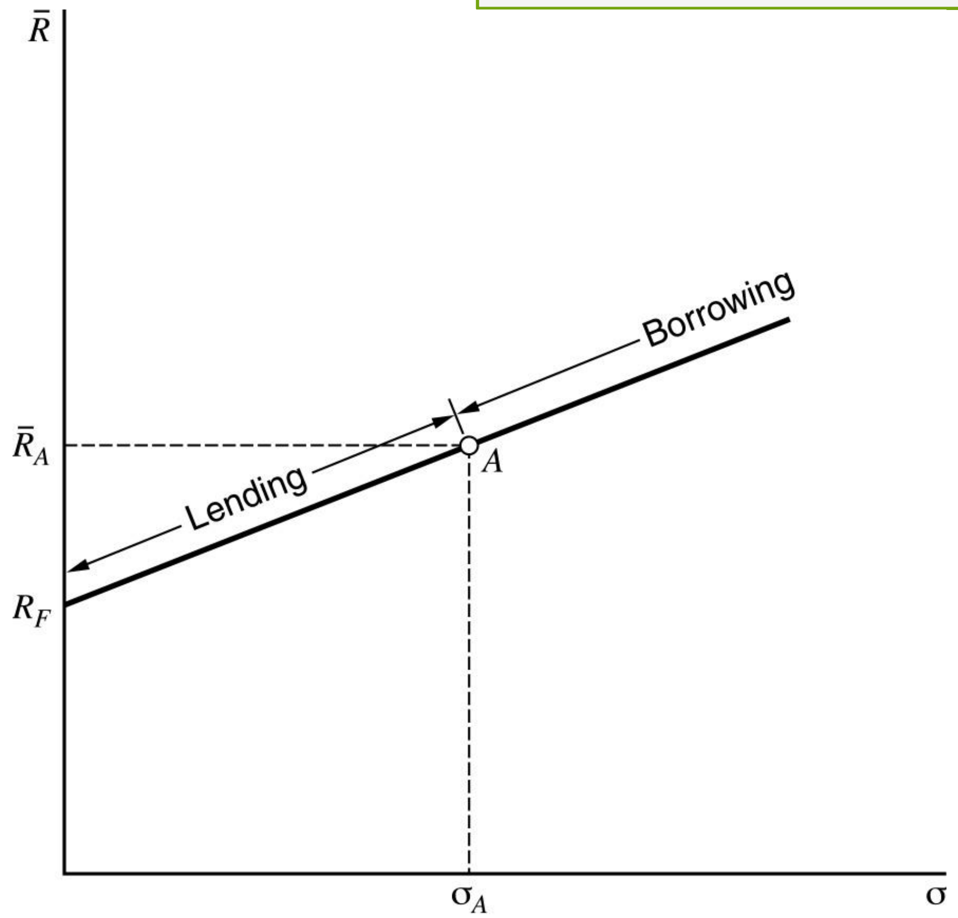

FIGURE 5-13 Expected return and risk when the risk-free rate is mixed with portfolio *A*.

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$$\overline{R}_p = X_c\overline{R}_c + X_s\overline{R}_s = X_c\overline{R}_c + (1 - X_c)\overline{R}_s$$

$$\sigma_p = (X_c^2\sigma_c^2 + X_s^2\sigma_s^2 + 2X_cX_s\rho_{cs}\sigma_c\sigma_s)^{1/2}$$

A fronteira eficiente com taxa de juro sem risco

- $\bar{R}_c = (1 - X_A)R_F + X_A\bar{R}_A$
- $\sigma_c = (X_A^2\sigma_A^2)^{1/2} = X_A\sigma_A \Rightarrow X_A = \dfrac{\sigma_c}{\sigma_A}$
- $\bar{R}_c = R_F + \left(\dfrac{\bar{R}_A - R_F}{\sigma_A}\right)\sigma_c$

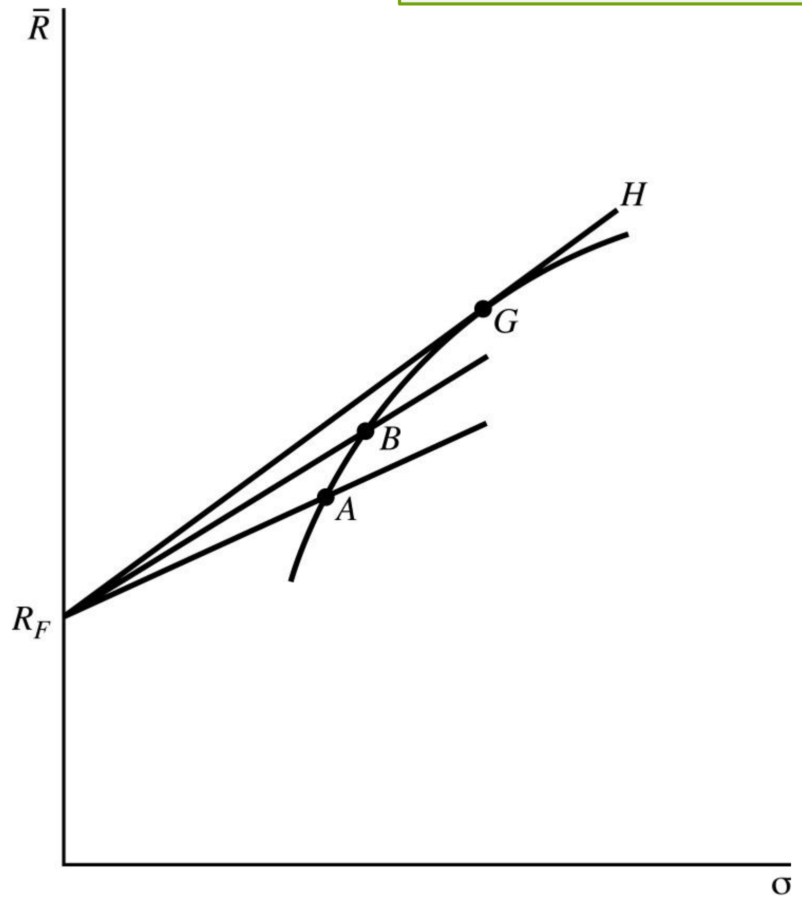

FIGURE 5-14 Combinations of the riskless asset and various risky portfolios.

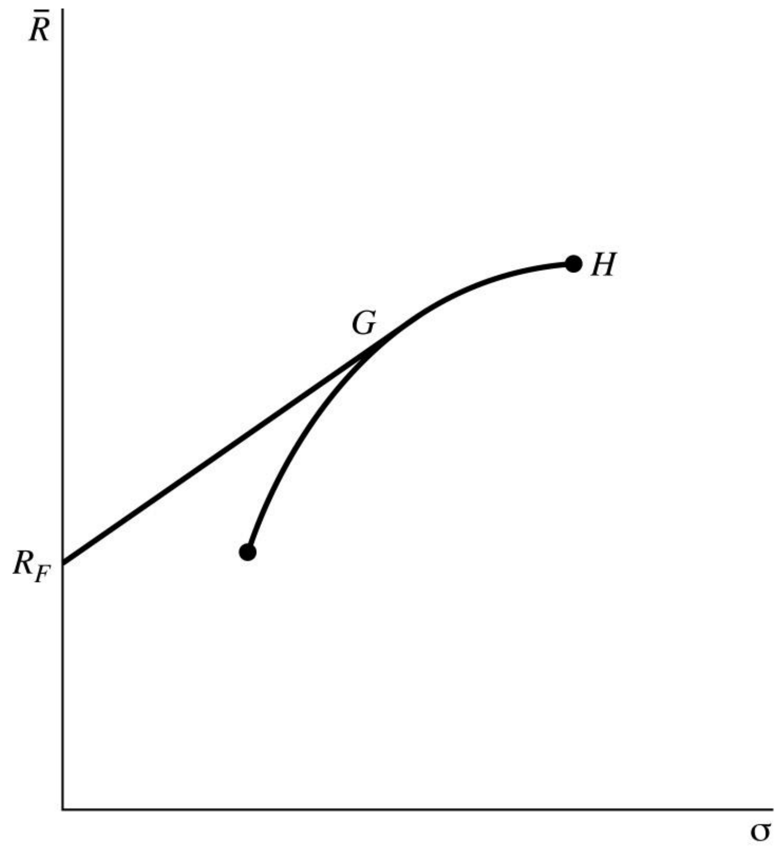

FIGURE 5-15 The efficient frontier with lending but not borrowing at the riskless rate.

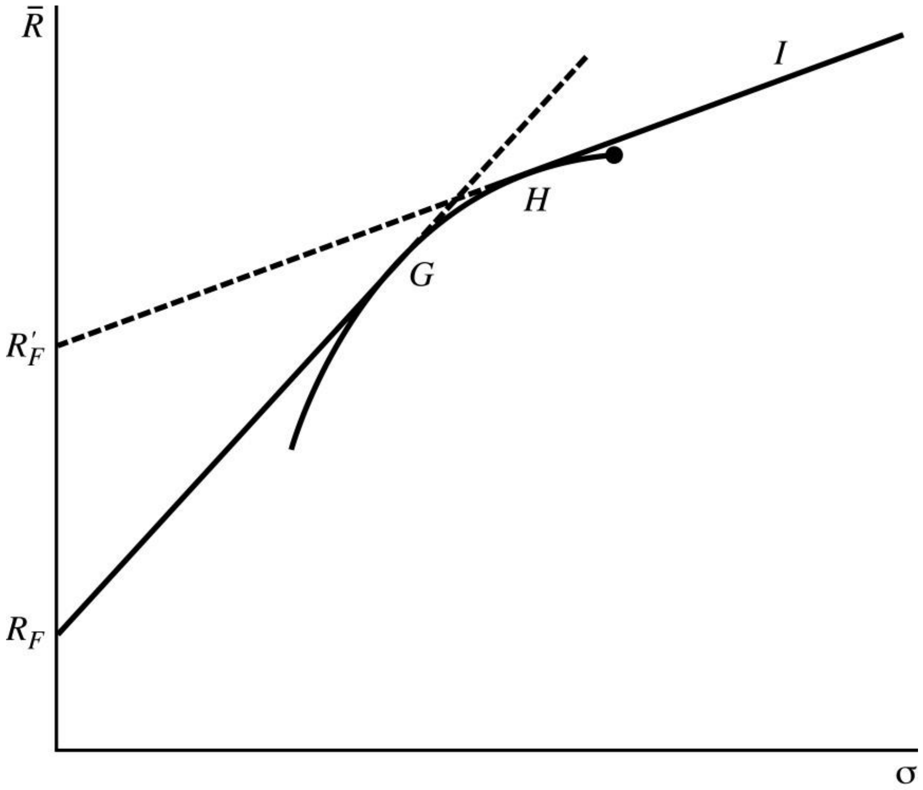

FIGURE 5-16 The efficient frontier with riskless lending and borrowing at different rates.

	Arithmetic Mean	Standard Deviation	S&P	Bonds	T-Bills	Inflation
			Correlations			
Stocks	10.03	15.67	1.00			
Bonds	5.73	11.14	0.22	1.00		
T-Bills	4.56	3.19	−0.15	0.24	1.00	
Inflation	4.24	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

Table 5-6 Returns with No Inflation Adjustment

	Arithmetic Mean	Standard Deviation	S&P	Bonds	T-Bills
S&P	5.78	17.32	1.00		
Bonds	1.49	12.39	0.37	1.00	
T-Bills	0.31	3.83	0.33	0.54	1.00

Table 5-7 Returns After Adjusting for Inflation

	Beginning Date	Arithmetic Mean Return Starting in 1926	Arithmetic Mean Return Starting in 1985	Standard Deviation Starting in 1926	Standard Deviation Starting in 1985
S&P 500	1926	10.82	17.15	22.03	17.89
U.S. Small Stocks	1926	12.36	14.46	35.33	22.69
U.S. Government Bonds	1926	5.32	11.98	8.08	10.44
IFC Emerging Market Index	1985	11.91	11.91	26.17	26.17

Source: Courtesy of Ibbotson Associates.

Table 5-8 Risk and Return Over Different Horizons

Top Triangle: All Periods	S&P 500	Small	Bonds	IFC
S&P 500	1.00	0.83	0.18	0.43
Small	0.67	1.00	0.09	0.46
Bonds	0.30	0.07	1.00	−0.15
IFC	0.43	0.46	−0.15	1.00
Bottom Triangle Common	S&P 500	Small	Bonds	IFC

Source: Courtesy of Ibbotson Associates.

Table 5-9 Correlation over Different Horizons

Time Period 1926–1991	Annualized Arithmetic Mean Return Based on Annualized 10-Year Returns	Arithmetic Mean Return Based on Annual Returns	Standard Deviation Based on Annualized 10-Year Returns	Standard Deviation Based on Annual Returns
S&P 500	9.8%	9.6%	20.7%	19.9%
U.S. Government Bonds	4.3%	4.8%	6.1%	7.7%
Treasury Bills	3.6%	3.6%	6.0%	3.2%

Source: F. Edwards and W. Goetzmann [11].

Table 5-10 The Effect of Time Horizon on Risk

Top Triangle: 10-Year	S&P	Bonds	T-Bills
S&P 500	1.00	0.06	0.19
Bonds	0.14	1.00	0.08
T-Bills	−0.03	0.22	1.00
Bottom Triangle: 1-Year			

Table 5-11 Correlations over Different Time Horizons

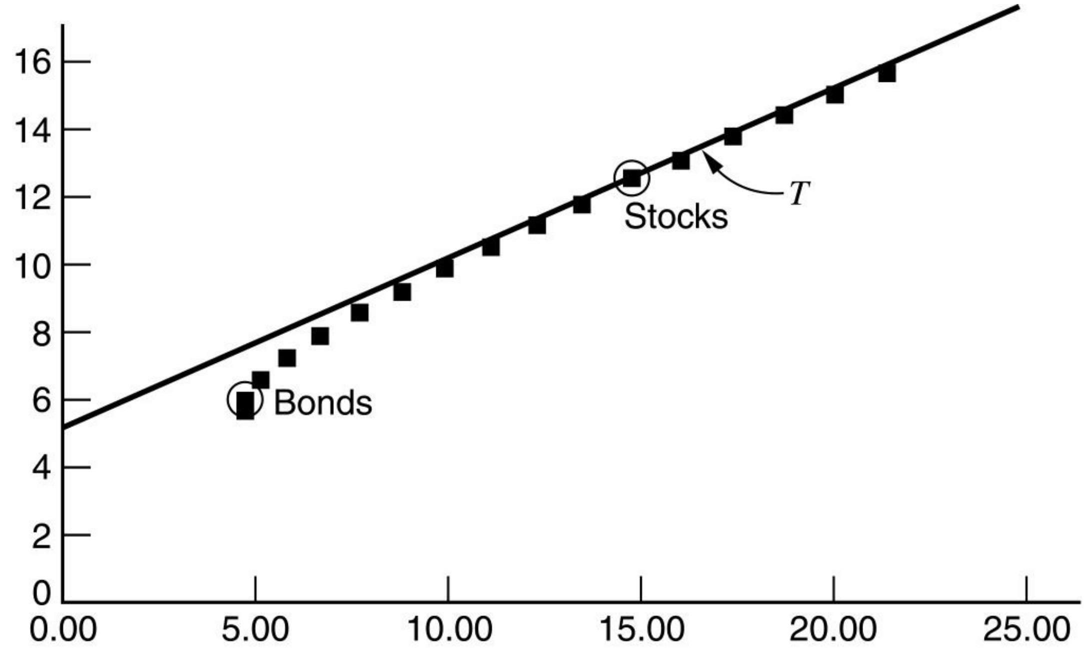

FIGURE 5-17 The efficient frontier.

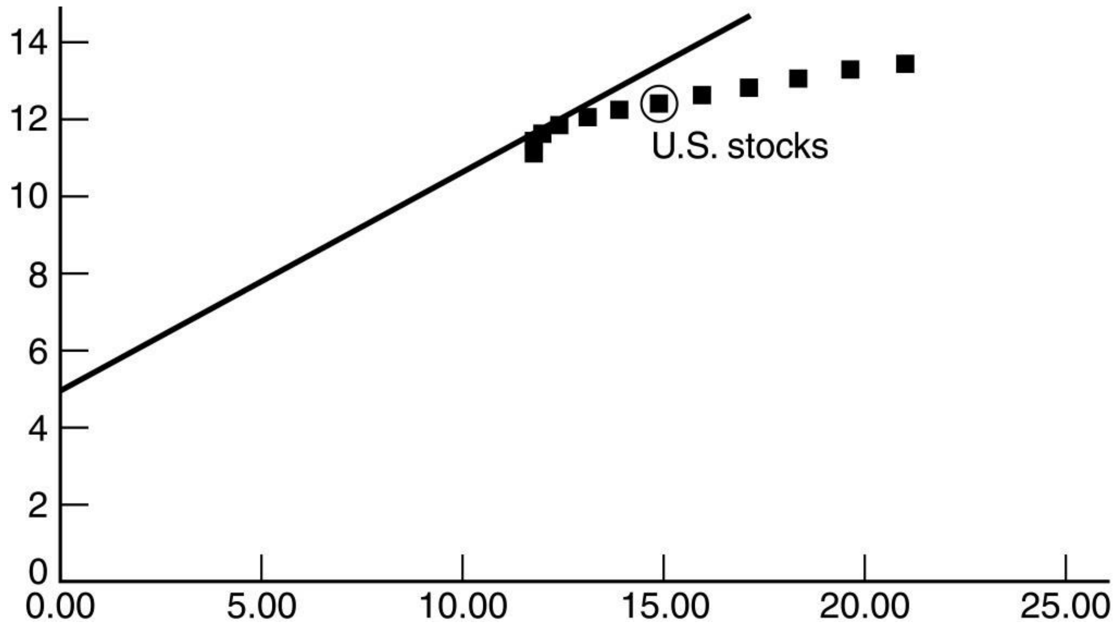

FIGURE 5-18 The efficient frontier.

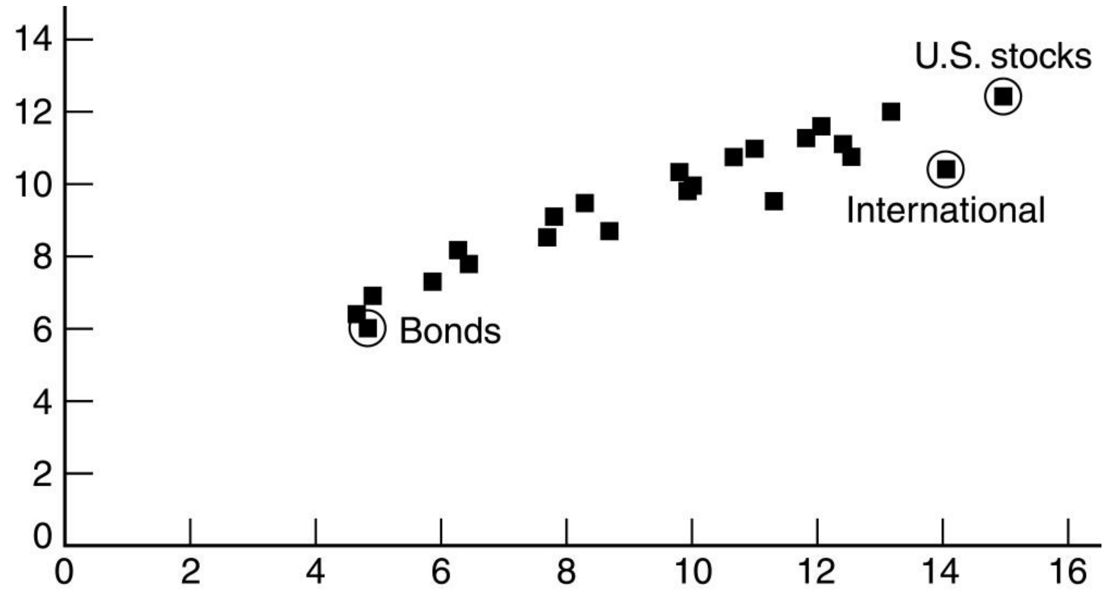

FIGURE 5-19 Combinations of bonds, domestic stocks, and international stocks.

Principais tipos de ordem

- Ordem de mercado comum (compra e venda).
- Ordem de limite de posição (limit order).
- Venda a descoberto (short sale).
- Ordem de cancelamento (stop order).

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- Técnicas para identificar a fronteira eficiente.

Vamos ver três alternativas

- 1- Que as vendas a descoberto são permitidas, e que é possível emprestar e pedir emprestado à taxa de juro sem risco.
- 2- Que as vendas a descoberto não são permitidas, mas é possível emprestar e pedir emprestado à taxa de juro sem risco.
- 3- Que não são permitidas vendas a descoberto nem utilizar a taxa de juro sem risco.
- 4- Que as vendas a descoberto são permitidas, e é possível emprestar ou pedir emprestado à taxa de juro sem risco, mas existem restrições quanto a dividendos.

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- 1- Que as vendas a descoberto são permitidas, e que é possível emprestar e pedir emprestado à taxa de juro sem risco.

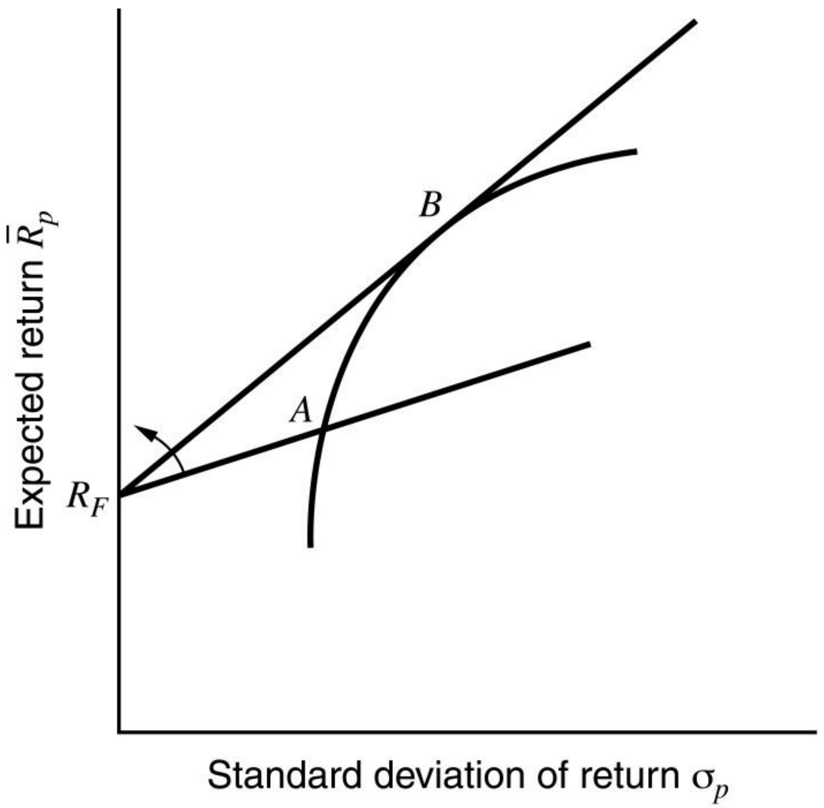

FIGURE 6-1 Combinations of the riskless asset in a risky portfolio.

$$Max \quad \theta = \frac{\overline{R}_P - R_F}{\sigma_P}$$

$$Suj.\,a: \quad \sum_{i=1}^{N} X_i = 1$$

$$R_F = 1R_F = \left(\sum_{i=1}^{N} X_i \right) R_F = \sum_{i=1}^{N} \left(X_i R_F \right)$$

$$\theta = \frac{\sum_{i=1}^{N} X_i\left(\bar{R}_i - R_F\right)}{\left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} X_i X_j \sigma_{ij}\right]^{1/2}}$$

1. $$\frac{d\theta}{dX_1} = 0$$

2. $$\frac{d\theta}{dX_2} = 0$$

3. $$\frac{d\theta}{dX_3} = 0$$

$\vdots$

N. $$\frac{d\theta}{dX_N} = 0$$

$$\lambda = \frac{\bar{R}_P - R_F}{\sigma_P^2} =$$

$$\frac{d\theta}{dX_i} = -\left(\lambda X_1 \sigma_{1i} + \lambda X_2 \sigma_{2i} + \lambda X_3 \sigma_{3i} + \cdots + \lambda X_i \sigma_i^2 + \cdots \right.$$

$$\left. + \lambda X_{N-1} \sigma_{N-1\,i} + \lambda X_N \sigma_{Ni} \right) + \bar{R}_i - R_F = 0$$

$$\begin{cases} R_1 - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + \ldots + Z_N\sigma_{1N} \\ \vdots \\ R_N - R_F = Z_1\sigma_{N1} + Z_2\sigma_{N2} + \ldots + Z_N\sigma_N^2 \end{cases}$$

$$X_i = \frac{Z_i}{\sum_{i=1}^{N} Z_i}$$

$$Max \quad \theta = \frac{\overline{R}_P - R_F}{\sigma_P}$$

$$Suj.a: \quad \sum_{i=1}^{N} X_i = 1$$

$$\begin{cases} R_1 - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + \ldots + Z_N\sigma_{1N} \\ \vdots \\ R_N - R_F = Z_1\sigma_{N1} + Z_2\sigma_{N2} + \ldots + Z_N\sigma_N^2 \end{cases}$$

$$X_i = \frac{Z_i}{\sum_{i=1}^{N} Z_i}$$

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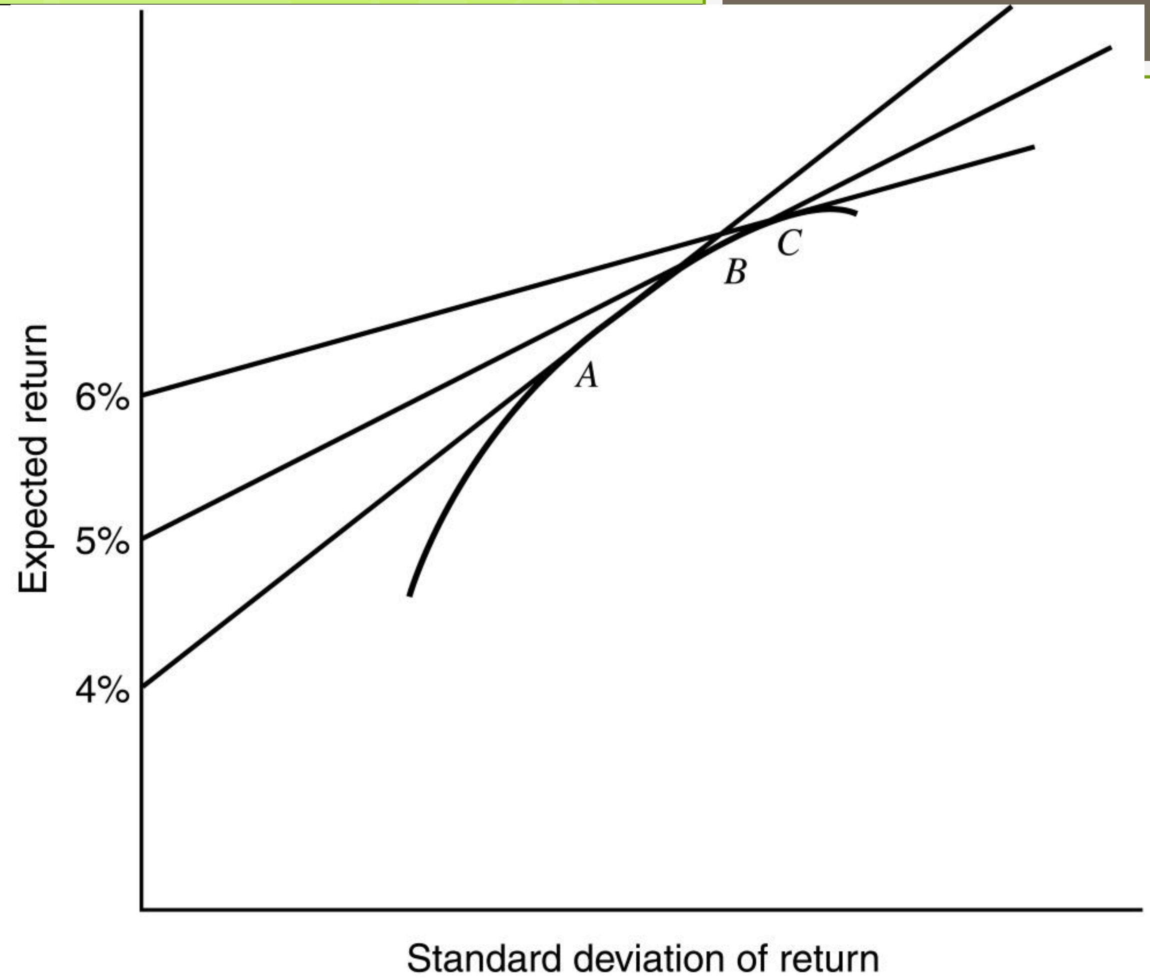

FIGURE 6-3 Tangency portfolios for different riskless rates.

- 2- Que as vendas a descoberto não são permitidas, mas é possível emprestar e pedir emprestado à taxa de juro sem risco.

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$$Max \quad \theta = \frac{\overline{R}_P - R_F}{\sigma_P}$$

$$Suj.a: \quad \sum_{i=1}^{N} X_i = 1 \quad \wedge \quad X_i > 0$$

- 3- Que as vendas a descoberto não são permitidas, nem é possível emprestar ou pedir emprestado à taxa de juro sem risco.

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$$\text{Minimize} \sum_{i=1}^{N}\left(X_i^2\sigma_i^2\right)+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left(X_iX_j\sigma_{ij}\right)$$

Subject to

$$(1)\quad \sum_{i=1}^{N} X_i = 1$$

$$(2)\quad \sum_{i=1}^{N}\left(X_i\bar{R}_i\right)=\bar{R}_P$$

$$(3)\quad X_i \geq 0, \qquad i=1,\ldots,N$$

- 4- Que as vendas a descoberto são permitidas, e é possível emprestar ou pedir emprestado à taxa de juro sem risco, mas existem restrições quanto a dividendos.

$$Max \quad \theta = \frac{\overline{R}_P - R_F}{\sigma_P}$$

$$Suj.a: \quad \sum_{i=1}^{N} X_i = 1$$

$$\sum_{i=1}^{N} X_i d_i \geq D$$

	S&P	Bonds	Canadian	Japan	Emerging Market	Pacific	Europe	Small Stock
Expected return	14.00	6.50	11.00	14.00	16.00	18.00	12.00	17.00
Standard deviation	18.50	5.00	16.00	23.00	30.00	26.00	20.00	24.00
				Correlation Coefficients				
S&P	1.00	0.45	0.70	0.20	0.64	0.30	0.61	0.79
Bonds		1.00	0.27	−0.01	0.41	0.01	0.13	0.28
Canadian			1.00	0.14	0.51	0.29	0.48	0.59
Japan				1.00	0.25	0.73	0.56	0.13
Emerging Market					1.00	0.28	0.61	0.75
Pacific						1.00	0.54	0.16
Europe							1.00	0.44
Small stock								1.00

Table 6-1 Input Data for Asset Allocation

Short sales allowed

$$\bar{R}_p = 5 + 0.714\sigma_p$$

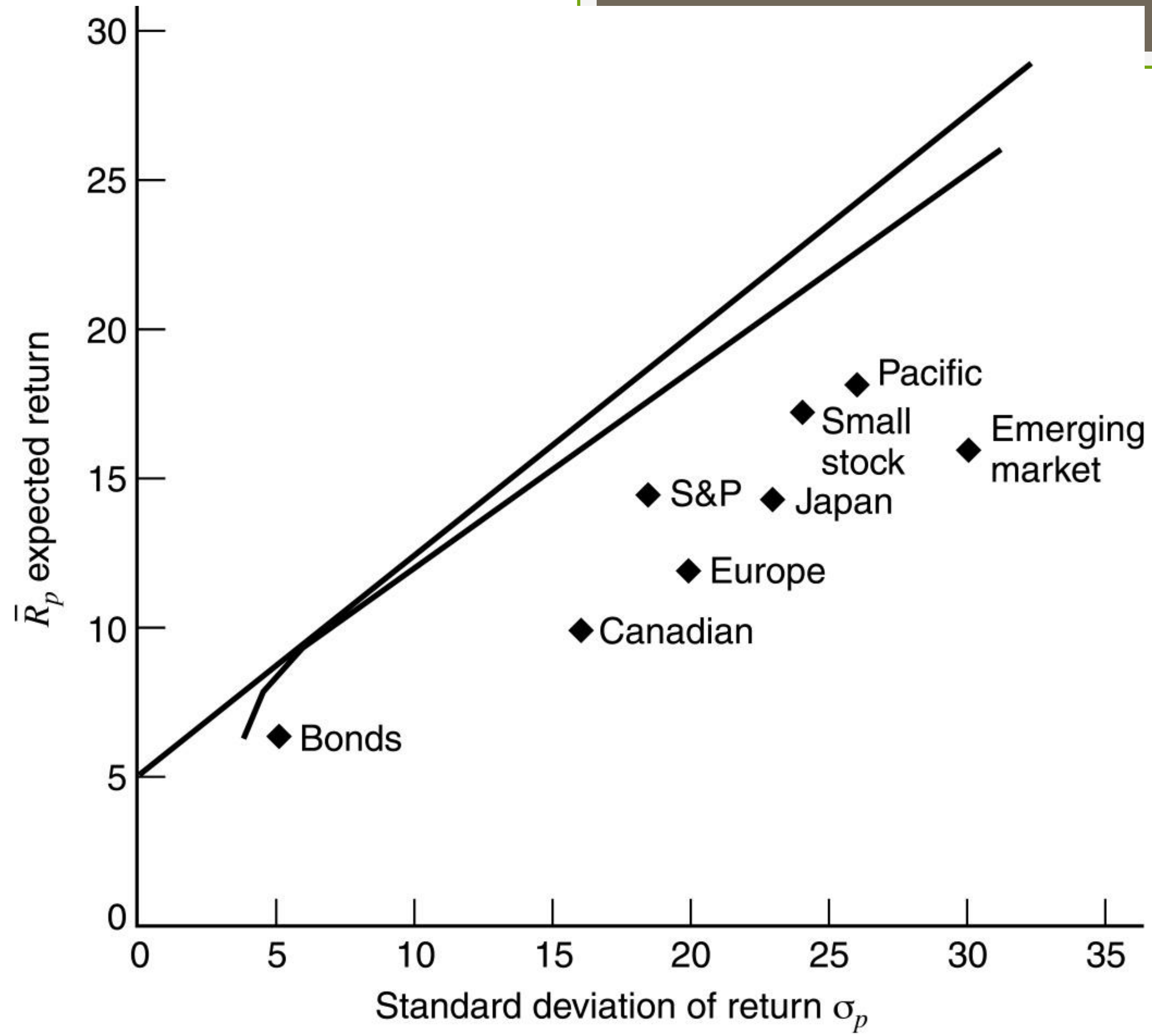

FIGURE 6-4 The efficient frontier with riskless lending and borrowing and short sales allowed.

	Global Minimum	1	2	3	4	5
Mean return	6.89	9.36	11.83	14.30	16.77	18.00
Standard deviation	4.88	6.66	10.03	13.86	17.87	26.00
			Proportions			
S&P	0.00	0.00	0.00	0.00	0.63	0.00
Bond	95.16	72.91	50.51	28.12	5.51	0.00
Canadian	0.06	0.00	0.00	0.00	0.00	0.00
Japan	3.96	3.57	3.17	2.77	2.41	0.00
Emerging market	0.00	0.00	0.00	0.00	0.00	0.00
Pacific	0.81	12.42	22.86	33.29	43.62	100.00
Europe	0.00	0.00	0.00	0.00	0.00	0.00
Small stock	0.00	11.10	23.46	35.82	47.82	0.00

Table 6-2 Proportions Invested When Short Sales Not Allowed

Short sales not allowed

$$\bar{R}_p = 5 + 0.685_p$$

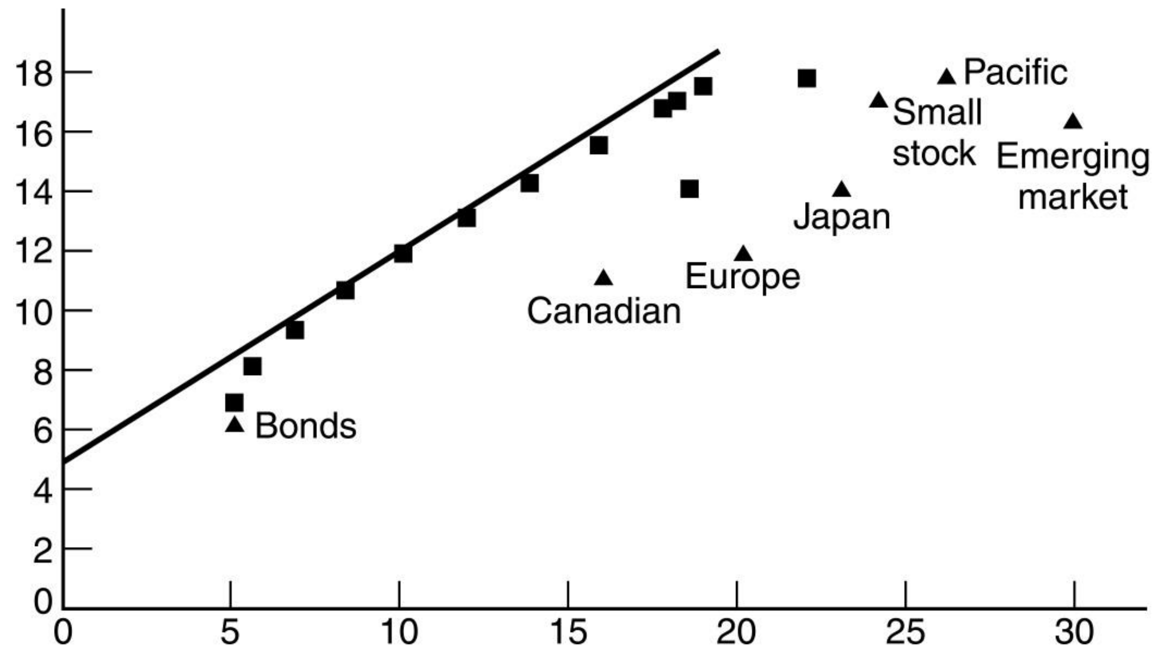

FIGURE 6-5 The efficient frontier with no riskless lending and borrowing and no short sales.

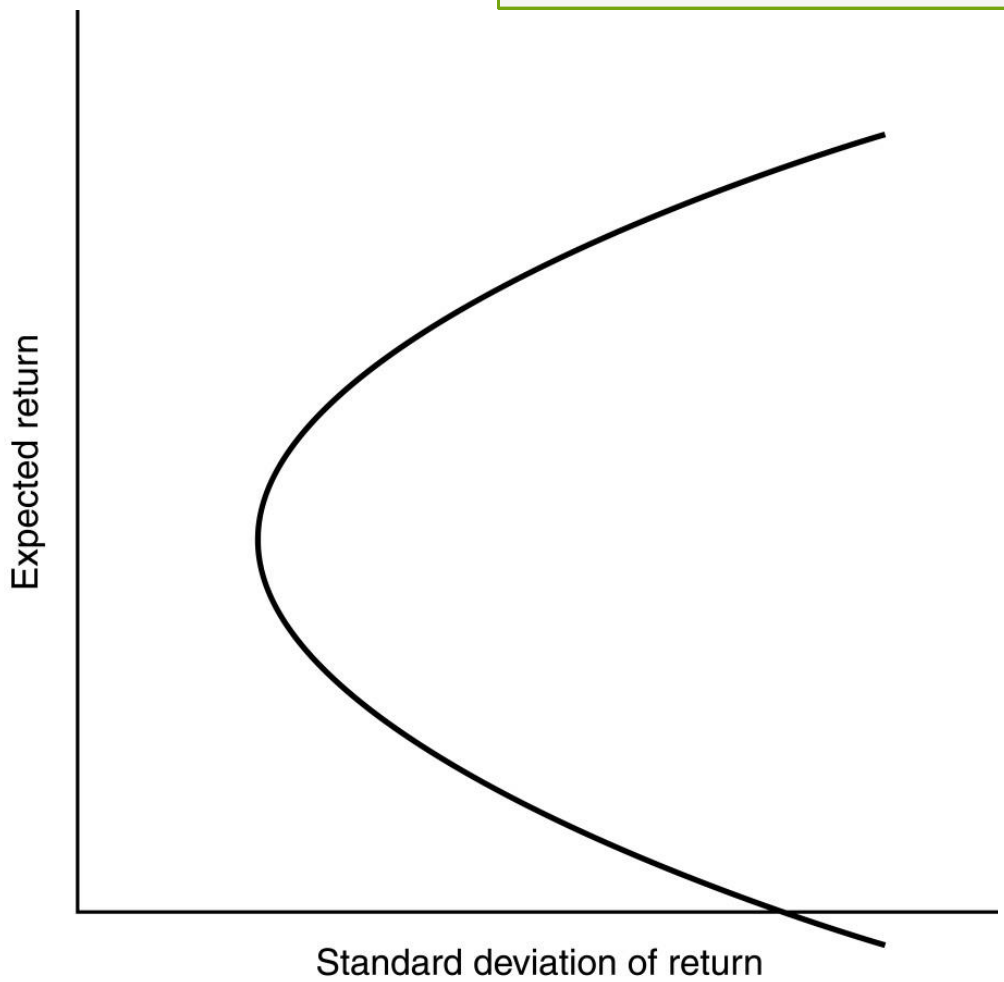

FIGURE 6-6 The minimum variance frontier.

- Identificação da estrutura de correlação dos rendimentos dos títulos financeiros.

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$$\bar{R}_p = E(R_p) = E\left(\sum_{i=1}^{N} X_i R_{ij}\right) = \sum_{i=1}^{N} X_i \bar{R}_i$$

$$\sigma_p^2 = \sum_{j=1}^{N} X_j^2 \sigma_j^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ j \neq k}}^{N} X_j X_k \rho_{jk} \sigma_j \sigma_k$$

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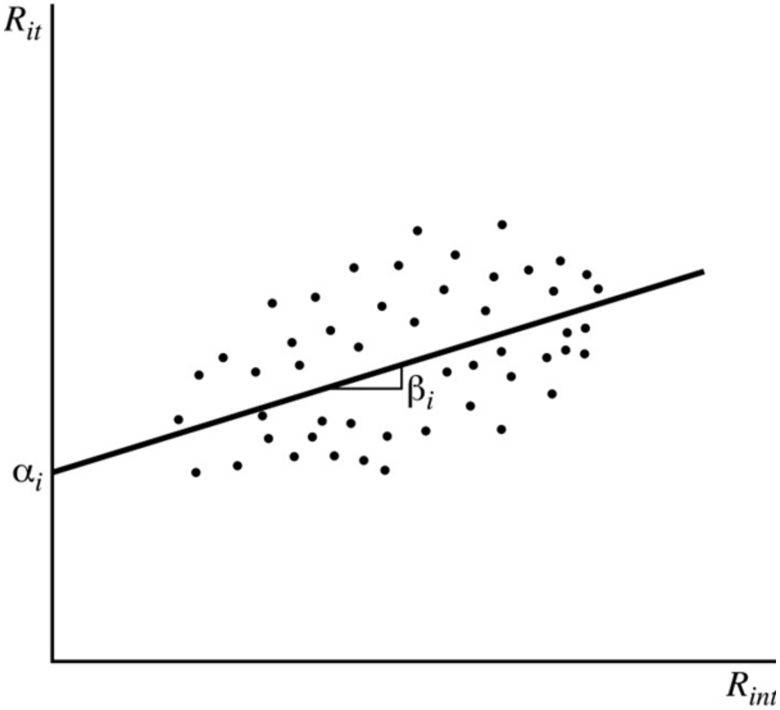
Single Index Model

$$R_i = a_i + \beta_i R_m$$

$$a_i = \alpha_i + e_i$$


R_{it}
α_i
β_i
R_{int}

$$R_i = \alpha_i + \beta_i R_m + e_i$$

$$\begin{cases} \operatorname{cov}(e_i e_j) = 0 \\ \mathrm{E}(e_i) = 0 \\ \operatorname{cov}(e_i R_m) = 0 \end{cases}$$

Propriedades esperadas

$$\overline{R}_i = \alpha_i + \beta_i \overline{R}_m$$

Como

$$\sigma_i^2 = E\left(R_i - \bar{R}_i\right)^2$$

Então

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$$

Como

$$\sigma_{ij} = E\left[\left(R_i - \bar{R}_i\right)\left(R_j - \bar{R}_j\right)\right]$$

Então

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2$$

Month	1 Return on Stock	2 Return on Market	3 R_i		4 α_i		5 $\beta_i R_m$	6 $+ e_i$
1	10	4	10	=	2	+	6	+ 2
2	3	2	3	=	2	+	3	− 2
3	15	8	15	=	2	+	12	+ 1
4	9	6	9	=	2	+	9	− 2
5	3	0	3	=	2	+	0	+ 1
	40	20	40		10		30	0

Table 7-1 Decomposition of Returns for the Single-Index Model

$$\overline{R}_p = E(R_p) = E\left(\sum_{i=1}^{N} X_i R_{ij}\right) = \sum_{i=1}^{N} X_i \overline{R}_i$$

$$\overline{R}_P = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \overline{R}_M$$

$$\sigma_p^2 = \sum_{j=1}^{N} X_j^2 \sigma_j^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ j \neq k}}^{N} X_j X_k \rho_{jk} \sigma_j \sigma_k$$

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

Características do single index model

$$\overline{R}_p = \alpha_p + \beta_p \overline{R}_m$$

Onde:

$$\beta_p = \sum_{i=1}^{N} X_i \beta_i$$

$$\alpha_p = \sum_{i=1}^{N} X_i \alpha_i$$

Do mesmo modo:

$$\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

Onde:

$$\beta_p = \sum_{i=1}^{N} X_i \beta_i$$

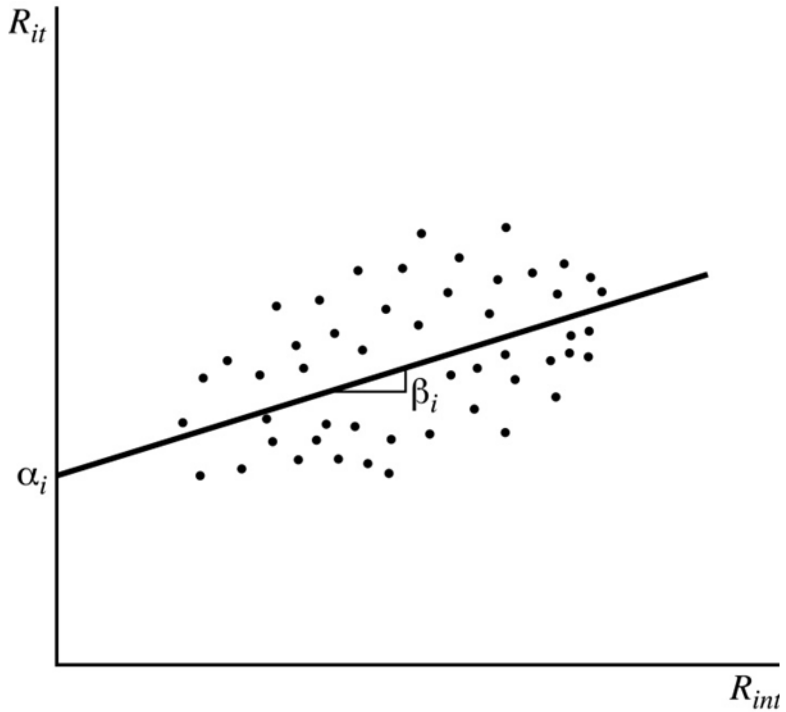
Number of Securities	Residual Risk (Variance) Expressed as a Percent of the Residual Risk Present in a One-Stock Portfolio with σ^2_{ei} a Constant
1	100
2	50
3	33
4	25
5	20
10	10
20	5
100	1
1000	0.1

Table 7-2 Residual Risk and Portfolio Size

Estimação dos Betas

Tal como vimos:

$$R_i = \alpha_i + \beta_i R_m + e_i$$



R_{it}
α_i
β_i
R_{int}

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

Supondo que n=60 (isto é 60 períodos).

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\sum_{t=1}^{60}\left[\left(R_{it} - \bar{R}_i\right)\left(R_{mt} - \bar{R}_m\right)\right]}{\sum_{t=1}^{60}\left(R_{mt} - \bar{R}_m\right)^2}$$

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$$\rho_{ij} = \frac{\beta_i \beta_j \sigma_m^2}{\sigma_i \sigma_j}$$

Precisão das estimativas de Beta.

Number of Securities in the Portfolio	Correlation Coefficient	Coefficient of Determination
1	0.60	0.36
2	0.73	0.53
4	0.84	0.71
7	0.88	0.77
10	0.92	0.85
20	0.97	0.95
35	0.97	0.95
50	0.98	0.96

Table 7-3 Association of Betas Over Time

- Alterações e ajustamentos do Beta ao longo do tempo.

Portfolio	7/54–6/61	7/61–6/68
1	0.393	0.620
2	0.612	0.707
3	0.810	0.861
4	0.987	0.914
5	1.138	0.995
6	1.337	1.169

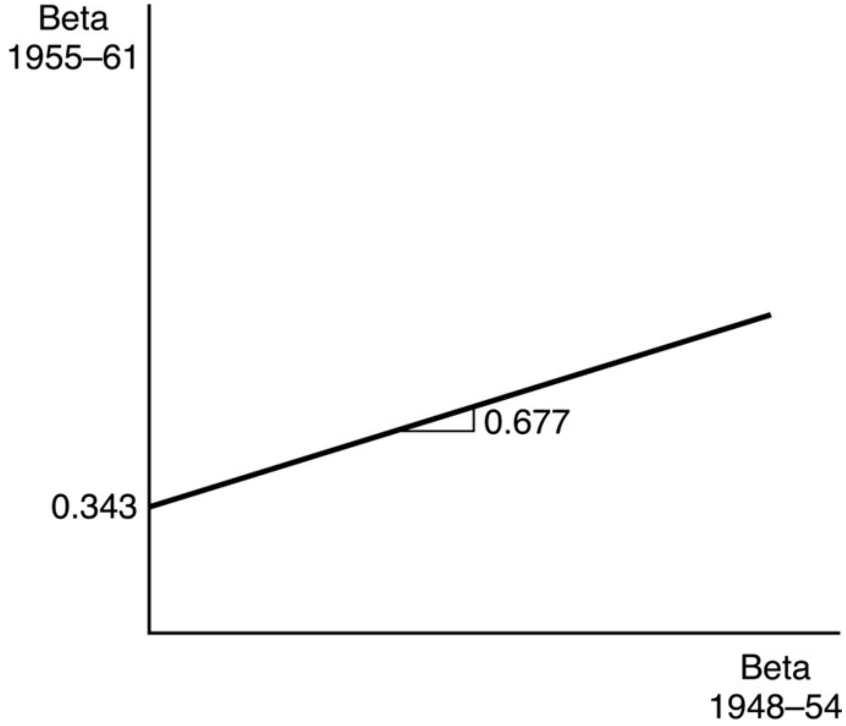
Source: Blume, Marchell. "On the Assessment of Risk," *Journal of Finance*, VI, No. 1 (March 1971).

Table 7-4 Betas on Ranked Portfolios for Two Successive Periods

Blume et al estudaram um modelo que relaciona os betas destes dois períodos. Os seus resultados foram

Modelo de Blume

- $\beta_{i2} = 0{,}343 + 0{,}677\,\beta_{i1}$


Beta
1955–61
0.677
0.343
Beta
1948–54

Modelo alternativo em Vasicek's para ajustar o Beta em relação ao Beta médio

Modelo de Vasicek

- $$\beta_{i2} = \frac{\sigma^2_{\beta_{i1}}}{\sigma^2_{\bar{\beta}_1} + \sigma^2_{\beta_{i1}}}\bar{\beta}_1 + \frac{\sigma^2_{\bar{\beta}_1}}{\sigma^2_{\bar{\beta}_1} + \sigma^2_{\beta_{i1}}}\beta_{i1}$$

Correcção do Beta de acordo com as variáveis financeiras da empresa.

Variable	Period 1 1947–1956 One-Stock Portfolio	Period 1 1947–1956 Five-Stock Portfolio	Period 2 1957–1965 One-Stock Portfolio	Period 2 1957–1965 Five-Stock Portfolio
Payout	−0.50	−0.77	−0.24	−0.45
Growth	0.23	0.51	0.03	0.07
Leverage	0.23	0.45	0.25	0.56
Liquidity	−0.13	−0.44	−0.01	−0.01
Size	−0.07	−0.13	−0.16	−0.30
Earnings variability	0.58	0.77	0.36	0.62
Earnings Beta	0.39	0.67	0.23	0.46

Table 7-5 Correlation between Accounting Measures of Risk and Market Beta

- Modelo de índices múltiplos.

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$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots + b_{iL}I_L + c_i$$

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$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + b_{i3}I_3 + \cdots + b_{iL}I_L + c_i$$

for all stocks $i = 1, ..., N$ (8.1)

BY DEFINITION

1. Residual variance of stock i equals σ_{ci}^2 where $i = 1, ..., N$.
2. Variance of index j equals σ_{Ij}^2 where $j = 1, ..., L$.

BY CONSTRUCTION

1. Mean of c_i equals $E(c_i) = 0$ for all stocks, where $i = 1, ..., N$.
2. Covariance between indexes j and k equals $E[(I_j - \bar{I}_j)(I_k - \bar{I}_k)] = 0$ for all indexes, where $j = 1, ..., L$ and $k = 1, ..., L$ $(j \neq k)$.
3. Covariance between the residual for stock i and index j equals $E[c_i(I_j - \bar{I}_j)] = 0$ for all stocks and indexes, where $i = 1, ..., N$ and $j = 1, ..., L$.

BY ASSUMPTION

1. Covariance between c_i and c_j is zero ($E(c_i c_j) = 0$) for all stocks where $i = 1, ..., N$ and $j = 1, ..., N$ $(j \neq i)$.

Retorno Esperado

$$R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + \ldots + b_{iL} I_L + c_i$$

$$\bar{R}_i = a_i + b_{i1} \bar{I}_{i2} + b_{i2} \bar{I}_{i2} + \ldots + b_{iL} \bar{I}_L$$

Variância do retorno

$$\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \ldots + b_{iL}^2\sigma_{IL}^2 + \sigma_{ci}^2$$

Covariância do retorno entre rendimentos.

$$\sigma_{ij} = b_{i1} b_{j1} \sigma_{I1}^2 + b_{i2} b_{j2} \sigma_{I2}^2 + \ldots + b_{iL} b_{jL} \sigma_{IL}^2$$

- Exemplo modelo da indústria que pressupões que o rendimento do título pode ser afectado pelo rendimento do mercado e pelo rendimento de industrias distintas.

$$R_i = a_i + b_{im} I_m + b_{i1} I_1 + b_{i2} I_2 + \ldots + b_{iL} I_L + c_i$$

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- Para uma empresa i na industria j a equação do retorno seria:

Para a indústria j:

$$\bar{R}_i = a_i + b_{im} I_m + b_{ij} I_j + c_i$$

$$\sigma_{ik} = b_{im} b_{km} \sigma_m^2 + b_{ij} b_{kj} \sigma_{Ij}^2 \quad \vee \quad \sigma_{ik} = b_{im} b_{km} \sigma_m^2$$

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- Modelos mistos.
- a) Fama and French
- b) Chen, Roll and Ross

a) Fama and French

$$E(R_i) = R_f + [E(R_m) - R_f]\,b_i + s_i E(SMB) + h_i E(HML) + \varepsilon_i$$

SMB é a diferença entre o portfólio das pequenas e grandes (big) empresas.

HML é a diferença entre o portfolio das empresas com market to book elevado (high) e as empresas com market to book baixo (low)

b) Chen, Roll and Ross

	I_1 Default	I_2 Term Structure	I_3 Deflation	I_4 Growth	I_5 Residual Market	R^2
Sector Name						
Cyclical	−1.63	0.55	2.84	−1.04	1.14	0.77
Growth	−2.08	0.58	3.16	−0.92	1.28	0.84
Stable	−1.40	0.68	2.31	−0.22[a]	0.74	0.73
Oil	−0.63[a]	0.31	2.19[a]	−0.83[a]	1.14	0.50
Utility	−1.06	0.72	1.54	0.23[a]	0.62	0.67
Transportation	−2.07	0.58	4.45	−1.13	1.37	0.66
Financial	−2.48	1.00	3.20	−0.56[a]	0.99	0.72

[a]Indicates *not* statistically different from zero at the 5% level.

Exemplo de outros possíveis factores

- Crescimento económico
- Fase do ciclo de negócio
- Crescimento na indústria
- Taxa de câmbio face a outras divisas.
- Etc.

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- Técnicas que simplificam o calculo da fronteira eficiente

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- O cálculo da fronteira eficiente seria simplificado se existisse um número concreto C que definisse se seria desejável incluir ou não um título no portfólio.

- Sabendo o excesso de rendimento de um título como uma proporção do beta:

$$\theta = \frac{(\overline{R}_i - R_F)}{\beta_i}$$

- O que haveria a fazer era:

1. Find the "excess return to Beta" ratio for each stock under consideration, and rank from highest to lowest.
2. The optimum portfolio consists of investing in all stocks for which $(\bar{R}_i - R_F)/\beta_i$ is greater than a particular cut-off point C^*. Shortly, we will define C^* and interpret its economic significance.

The preceding procedure is extremely simple. Once C^* has been determined, the securities to be included can be selected by inspection. Furthermore, the amount to invest in each security is equally simple to determine, as will be discussed shortly.

1	2	3	4	5	6
Security No. i	Mean Return $\bar{R}_i$	Excess Return $\bar{R}_i - R_F$	Beta β_i	Unsystematic Risk σ^2_{ei}	Excess Return over Beta $\frac{(\bar{R}_i - R_F)}{\beta_i}$
1	15	10	1	50	10
2	17	12	1.5	40	8
3	12	7	1	20	7
4	17	12	2	10	6
5	11	6	1	40	6
6	11	6	1.5	30	4
7	11	6	2	40	3
8	7	2	0.8	16	2.5
9	7	2	1	20	2
10	5.6	0.6	0.6	6	1.0

Table 9-1 Data Required to Determine Optimal Portfolio $R_F = 5\%$

- E como encontrar este ponto C*?

- $$C_i = \frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}}{1 + \sigma_m^2 \sum_{j=1}^{i} \left(\frac{\beta_j^2}{\sigma_{ej}^2} \right)} = \frac{\beta_{iP}(\bar{R}_P - R_F)}{\beta_i}$$

- P = carteira óptima,
- e = error term,
- i (ou j) é o título.
- O Beta de i e P = é a variação esperada no retorno do título que está associada com a variação de 1% no valor carteira óptima.

$$C^* = (\bar{R}_p - R_F)\frac{cov(i,p)}{{\sigma_P}^2}\beta_i$$

$$= \frac{\beta_{iP}(\bar{R}_P - R_F)}{\beta_i}$$

$$\frac{\bar{R}_i - R_F}{\beta_i} > C_i$$

1 Security No. i	2 $\frac{(\overline{R}_i - R_F)}{\beta_i}$	3 $\frac{(\overline{R}_i - R_F)\beta_i}{\sigma_{ei}^2}$	4 $\frac{\beta_i^2}{\sigma_{ei}^2}$	5 $\sum_{j=1}^{i} \frac{(\overline{R}_j - R_F)\beta_j}{\sigma_{ej}^2}$	6 $\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{ej}^2}$	7 C_i
1	10	2/10	2/100	2/10	2/100	1.67
2	8	4.5/10	5.625/100	6.5/10	7.625/100	3.69
3	7	3.5/10	5/100	10/10	12.625/100	4.42
4	6	24/10	40/100	34/10	52.625/100	5.43
5	6	1.5/10	2.5/100	35.5/10	55.125/100	5.45
6	4	3/10	7.5/100	38.5/10	62.625/100	5.30
7	3	3/10	10/100	41.5/10	72.625/100	5.02
8	2.5	1/10	4/100	42.5/10	76.625/100	4.91
9	2.0	1/10	5/100	43.5/10	81.625/100	4.75
10	1.0	0.6/10	6/100	44.1/10	87.625/100	4.52

Table 9-2 Calculations for Determining Cut-off Rate with $s^2_m = 10$

$$C_i = \frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\overline{R}_j - R_F)\beta_j}{\sigma_{ej}^2}}{1 + \sigma_m^2 \sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{ej}^2}}$$

$$= \frac{\sigma_m^2(\text{column } 5)}{1 + \sigma_m^2(\text{column } 6)} = \frac{10\left(\frac{2}{10}\right)}{1 + 10\left(\frac{2}{100}\right)} = 1.67$$

$$C_2 = \frac{\sigma_m^2 (\text{column } 5)}{1 + \sigma_m^2 (\text{column } 6)} = \frac{10 \frac{6.5}{10}}{1 + 10 \frac{7.625}{100}} = 3.68$$

1 Security No. i	2 $\frac{(\bar{R}_i - R_F)}{\beta_i}$	3 $\frac{(\bar{R}_i - R_F)\beta_i}{\sigma^2_{ei}}$	4 $\frac{\beta^2_i}{\sigma^2_{ei}}$	5 $\sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma^2_{ej}}$	6 $\sum_{j=1}^{i} \frac{\beta^2_j}{\sigma^2_{ej}}$	7 C_i
1	10	2/10	2/100	2/10	2/100	1.67
2	8	4.5/10	5.625/100	6.5/10	7.625/100	3.69
3	7	3.5/10	5/100	10/10	12.625/100	4.42
4	6	24/10	40/100	34/10	52.625/100	5.43
5	6	1.5/10	2.5/100	35.5/10	55.125/100	5.45
6	4	3/10	7.5/100	38.5/10	62.625/100	5.30
7	3	3/10	10/100	41.5/10	72.625/100	5.02
8	2.5	1/10	4/100	42.5/10	76.625/100	4.91
9	2.0	1/10	5/100	43.5/10	81.625/100	4.75
10	1.0	0.6/10	6/100	44.1/10	87.625/100	4.52

Table 9-2 Calculations for Determining Cut-off Rate with $s^2_m = 10$

- $C^* = C(4) = 5.45$

1 Security No. i	2 $\frac{(\bar{R}_i - R_F)}{\beta_i}$	3 $\frac{(\bar{R}_i - R_F)\beta_i}{\sigma_{ei}^2}$	4 $\frac{\beta_i^2}{\sigma_{ei}^2}$	5 $\sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}$	6 $\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{ej}^2}$	7 C_i
1	10	2/10	2/100	2/10	2/100	1.67
2	8	4.5/10	5.625/100	6.5/10	7.625/100	3.69
3	7	3.5/10	5/100	10/10	12.625/100	4.42
4	6	24/10	40/100	34/10	52.625/100	5.43
5	6	1.5/10	2.5/100	35.5/10	55.125/100	5.45
6	4	3/10	7.5/100	38.5/10	62.625/100	5.30
7	3	3/10	10/100	41.5/10	72.625/100	5.02
8	2.5	1/10	4/100	42.5/10	76.625/100	4.91
9	2.0	1/10	5/100	43.5/10	81.625/100	4.75
10	1.0	0.6/10	6/100	44.1/10	87.625/100	4.52

Table 9-2 Calculations for Determining Cut-off Rate with $s^2_m = 10$

- Falta contudo ainda definir qual a proporção do peso de cada título na carteira eficiente.

$$X_i = \frac{Z_i}{\sum_{incluidos} Z_j}$$

$$Z_i = \frac{\beta_i}{\sigma_{ei}^2}\left(\frac{\overline{R}_i - R_F}{\beta_i} - C^*\right)$$

$$Z_1 = \frac{2}{100}(10 - 5.45) = 0.091$$

$$Z_2 = \frac{3.75}{100}(8 - 5.45) = 0.095625$$

$$Z_3 = \frac{5}{100}(7 - 5.45) = 0.0775$$

$$Z_4 = \frac{20}{100}(6 - 5.45) = 0.110$$

$$Z_5 = \frac{2.5}{100}(6 - 5.45) = 0.01375$$

$$\sum_{i=1}^{5} Z_i = 0.387875$$

$$X_1 = \frac{0.091}{0.387875} = 23.5\%$$

$$X_2 = \frac{0.095625}{0.387875} = 24.6\%$$

$$X_3 = \frac{0.0775}{0.387875} = 20\%$$

$$X_4 = \frac{0.110}{0.387875} = 28.4\%$$

$$X_5 = \frac{0.01375}{0.387875} = 3.5\%$$

- Outro exemplo

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1 Security Number i	2 Mean Return $\bar{R}_i$	3 Excess Return $\bar{R}_i - R_F$	4 Beta β_i	5 Unsystematic Risk σ^2_{ei}	6 Excess Return over Beta $\frac{(\bar{R}_i - R_F)}{\beta_i}$
1	19	14	1.0	20	14
2	23	18	1.5	30	12
3	11	6	0.5	10	12
4	25	20	2.0	40	10
5	13	8	1.0	20	8
6	9	4	0.5	50	8
7	14	9	1.5	30	6
8	10	5	1.0	50	5
9	9.5	4.5	1.0	50	4.5
10	13	8	2.0	20	4
11	11	6	1.5	30	4
12	8	3	1.0	20	3
13	10	5	2.0	40	2.5
14	7	2	1.0	20	2

Table 9-3 Data Required to Determine Optimal Portfolio; $R_F = 10$

Security Number i	$\frac{(\bar{R}_i - R_F)}{\beta_i}$	$\frac{(\bar{R}_i - R_F)\beta_i}{\sigma^2_{ei}}$	$\frac{\beta^2_i}{\sigma^2_{ei}}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma^2_{ej}}$	$\sum_{j=1}^{i} \frac{\beta^2_j}{\sigma^2_{ej}}$	C_i
1	14	$\frac{70}{100}$	$\frac{5}{100}$	$\frac{70}{100}$	$\frac{5}{100}$	4.67
2	12	$\frac{90}{100}$	$\frac{7.5}{100}$	$\frac{160}{100}$	$\frac{12.5}{100}$	7.11
3	12	$\frac{30}{100}$	$\frac{2.5}{100}$	$\frac{190}{100}$	$\frac{15}{100}$	7.6
4	10	$\frac{100}{100}$	$\frac{10}{100}$	$\frac{290}{100}$	$\frac{25}{100}$	8.29
5	8	$\frac{40}{100}$	$\frac{5}{100}$	$\frac{330}{100}$	$\frac{30}{100}$	8.25
6	8	$\frac{4}{100}$	$\frac{0.5}{100}$	$\frac{334}{100}$	$\frac{30.5}{100}$	8.25
7	6	$\frac{45}{100}$	$\frac{7.5}{100}$	$\frac{379}{100}$	$\frac{38}{100}$	7.9

Table 9-4 Calculations for Determining Cut-off Rate with $s^2_m = 10$

Security Number i	$\frac{(\bar{R}_i - R_F)}{\beta_i}$	$\frac{(\bar{R}_i - R_F)\beta_i}{\sigma_{ei}^2}$	$\frac{\beta_i^2}{\sigma_{ei}^2}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}$	$\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{ej}^2}$	C_i
8	5	$\frac{10}{100}$	$\frac{2}{100}$	$\frac{389}{100}$	$\frac{40}{100}$	7.78
9	4.5	$\frac{9}{100}$	$\frac{2}{100}$	$\frac{398}{100}$	$\frac{42}{100}$	7.65
10	4	$\frac{80}{100}$	$\frac{20}{100}$	$\frac{478}{100}$	$\frac{62}{100}$	6.64
11	4	$\frac{30}{100}$	$\frac{7.5}{100}$	$\frac{508}{100}$	$\frac{69.5}{100}$	6.39
12	3	$\frac{15}{100}$	$\frac{5}{100}$	$\frac{523}{100}$	$\frac{74.5}{100}$	6.19
13	2.5	$\frac{25}{100}$	$\frac{10}{100}$	$\frac{548}{100}$	$\frac{84.5}{100}$	5.8
14	2	$\frac{10}{100}$	$\frac{5}{100}$	$\frac{558}{100}$	$\frac{89.5}{100}$	5.61

- $$C_i = \frac{\sigma_m^2 \sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}}{1 + \sigma_m^2 \sum_{j=1}^{i} \left(\frac{\beta_j^2}{\sigma_{ej}^2} \right)} = \frac{\beta_{iP}(\bar{R}_P - R_F)}{\beta_i}$$

- Onde P é a carteira óptima, e é o error term, i (ou j) é o título. O Beta de i e P é a variação esperada no retorno do título associada com a variação de 1% na carteira óptima.

Security Number i	$\frac{(\bar{R}_i - R_F)}{\beta_i}$	$\frac{(\bar{R}_i - R_F)\beta_i}{\sigma_{ei}^2}$	$\frac{\beta_i^2}{\sigma_{ei}^2}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma_{ej}^2}$	$\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{ej}^2}$	C_i
1	14	$\frac{70}{100}$	$\frac{5}{100}$	$\frac{70}{100}$	$\frac{5}{100}$	4.67
2	12	$\frac{90}{100}$	$\frac{7.5}{100}$	$\frac{160}{100}$	$\frac{12.5}{100}$	7.11
3	12	$\frac{30}{100}$	$\frac{2.5}{100}$	$\frac{190}{100}$	$\frac{15}{100}$	7.6
4	10	$\frac{100}{100}$	$\frac{10}{100}$	$\frac{290}{100}$	$\frac{25}{100}$	8.29
5	8	$\frac{40}{100}$	$\frac{5}{100}$	$\frac{330}{100}$	$\frac{30}{100}$	8.25
6	8	$\frac{4}{100}$	$\frac{0.5}{100}$	$\frac{334}{100}$	$\frac{30.5}{100}$	8.25
7	6	$\frac{45}{100}$	$\frac{7.5}{100}$	$\frac{379}{100}$	$\frac{38}{100}$	7.9

Table 9-4 Calculations for Determining Cut-off Rate with $s^2_m = 10$

Security Number i	$\frac{(\bar{R}_i - R_F)}{\beta_i}$	$\frac{(\bar{R}_i - R_F)\beta_i}{\sigma^2_{ei}}$	$\frac{\beta^2_i}{\sigma^2_{ei}}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_F)\beta_j}{\sigma^2_{ej}}$	$\sum_{j=1}^{i} \frac{\beta^2_j}{\sigma^2_{ej}}$	C_i
8	5	$\frac{10}{100}$	$\frac{2}{100}$	$\frac{389}{100}$	$\frac{40}{100}$	7.78
9	4.5	$\frac{9}{100}$	$\frac{2}{100}$	$\frac{398}{100}$	$\frac{42}{100}$	7.65
10	4	$\frac{80}{100}$	$\frac{20}{100}$	$\frac{478}{100}$	$\frac{62}{100}$	6.64
11	4	$\frac{30}{100}$	$\frac{7.5}{100}$	$\frac{508}{100}$	$\frac{69.5}{100}$	6.39
12	3	$\frac{15}{100}$	$\frac{5}{100}$	$\frac{523}{100}$	$\frac{74.5}{100}$	6.19
13	2.5	$\frac{25}{100}$	$\frac{10}{100}$	$\frac{548}{100}$	$\frac{84.5}{100}$	5.8
14	2	$\frac{10}{100}$	$\frac{5}{100}$	$\frac{558}{100}$	$\frac{89.5}{100}$	5.61

Supor que encontramos o seguinte ponto critico.

$$C^* = C_4 = \frac{58}{7} = 8{,}29$$

1 Security Number i	2 Mean Return $\bar{R}_i$	3 Excess Return $\bar{R}_i - R_F$	4 Beta β_i	5 Unsystematic Risk σ^2_{ei}	6 Excess Return over Beta $\frac{(\bar{R}_i - R_F)}{\beta_i}$
1	19	14	1.0	20	14
2	23	18	1.5	30	12
3	11	6	0.5	10	12
4	25	20	2.0	40	10
5	13	8	1.0	20	8
6	9	4	0.5	50	8
7	14	9	1.5	30	6
8	10	5	1.0	50	5
9	9.5	4.5	1.0	50	4.5
10	13	8	2.0	20	4
11	11	6	1.5	30	4
12	8	3	1.0	20	3
13	10	5	2.0	40	2.5
14	7	2	1.0	20	2

Table 9-3 Data Required to Determine Optimal Portfolio; $R_F = 10$

$$Z_i = \frac{\beta_i}{\sigma_{ei}^2}\left(\frac{\overline{R}_i - R_F}{\beta_i} - C^*\right)$$

$$Z_1 = \frac{1}{20}\left(14 - \frac{58}{7}\right) = \frac{40}{140} = \frac{240}{840}$$

$$Z_2 = \frac{1.5}{30}\left(12 - \frac{58}{7}\right) = \frac{39}{210} = \frac{156}{840}$$

$$Z_3 = \frac{0.5}{10}\left(12 - \frac{58}{7}\right) = \frac{13}{70} = \frac{156}{840}$$

$$Z_4 = \frac{2}{40}\left(10 - \frac{58}{7}\right) = \frac{24}{280} = \frac{72}{840}$$

$$X_i = \frac{Z_i}{\sum_{incluidos} Z_j}$$

$$X_1 = \frac{240}{240+156+156+72} = \frac{240}{624} = 0.38$$

$$X_2 = \frac{156}{240+156+156+72} = \frac{156}{624} = 0.25$$

$$X_3 = \frac{156}{240+156+156+72} = \frac{156}{624} = 0.25$$

$$X_4 = \frac{72}{240+156+156+72} = \frac{72}{624} = 0.12$$

- Como escolher o perfil de investimento?

- Como escolher o perfil de investimento?

- A análise neoclássica da teoria da utilidade

- Introdução às funções de preferencia.

Investment *A*		Investment *B*	
Outcome	Probability of Outcome	Outcome	Probability of Outcome
15	1/3	20	1/3
10	1/3	12	1/3
5	1/3	4	1/3

Table 10-1 Two Alternative Investments

	Islanders	Flyers
Wins	40	45
Ties	20	5
Losses	10	20

Table 10-2 Data for Ranking Hockey Teams

- Aplicando o cálculo à equipa dos Islanders:
- U= 40*2 + 20*1 + 10 * 0 = 100
- Para os Flyers:
- U = 45 * 2 + 5 *1 + 20 * 0 = 95

$$E(U) = \sum_{W=1}^{N} U(W)\, P(W)$$

Investment *A*		Investment *B*		Investment *C*	
Outcome	Probability	Outcome	Probability	Outcome	Probability
20	3/15	19	1/5	18	1/4
18	5/15	10	2/5	16	1/4
14	4/15	5	2/5	12	1/4
10	2/15			8	1/4
6	1/15				

Table 10-3 Outcomes and Associated Probabilities for Three Investments

- Supor a seguinte função de utilidade:
- $U(W)=4W-(1/10)W^2$

Investment *A*			Investment *B*			Investment *C*		
Outcome	Utility of Outcome	Probability	Outcome	Utility of Outcome	Probability	Outcome	Utility of Outcome	Probability
20	40	3/15	19	39.9	1/5	18	39.6	1/4
18	39.6	5/15	10	30	2/5	16	38.4	1/4
14	36.4	4/15	5	17.5	2/5	12	33.6	1/4
10	30	2/15				8	25.6	1/4
6	20.4	1/15						

Table 10-4 Including Utility

$$\text{Expected utility } A = (40)(3/15) + (39.6)(5/15) + (36.4)(4/15) + (30)(2/15) + (20.4)(1/15)$$

$$= \frac{544}{15} = 36.3$$

$$\text{Expected utility } B = (39.9)(1/5) + (30)(2/5) + (17.5)(2/5)$$

$$= \frac{134.9}{5} = 26.98$$

$$\text{Expected utility } C = (39.6)(1/4) + (38.4)(1/4) + (33.6)(1/4) + (25.6)(1/4)$$

$$= \frac{137.2}{4} = 34.4$$

- Propriedades económicas das funções de utilidade

Pressupostos:

- 1- Comparabilidade.
- 2- Transitividade.
- 3- Independência.
- 4- Valores de equivalência exactos.

Invest		Do Not Invest	
Outcome	Probability	Outcome	Probability
2	1/2	1	1
0	1/2		

- Primeira derivada > 0
- Segunda derivada define o perfil de risco do investidor.

Condition	Definition	Implication
1. Risk aversion	Reject fair gamble	$U''(0) < 0$
2. Risk neutrality	Indifferent to fair gamble	$U''(0) = 0$
3. Risk preference	Select a fair gamble	$U''(0) > 0$

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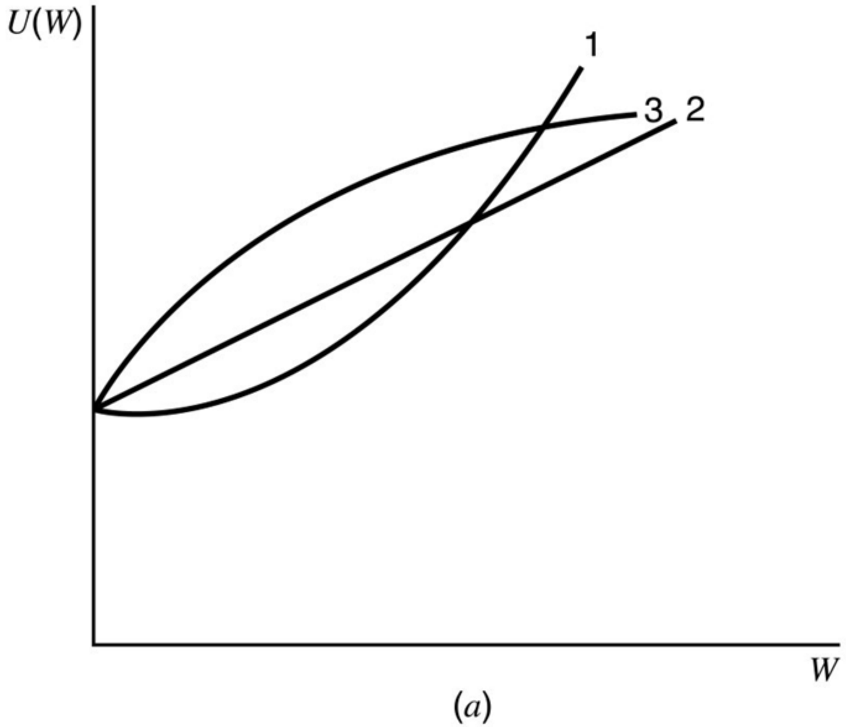

FIGURE 10-1 Characteristics of functions with different risk-aversion coefficients. (1) Utility function of a risk-seeking investor. (2) Utility function of a risk-neutral investor. (3) Utility function of a risk-averse investor.

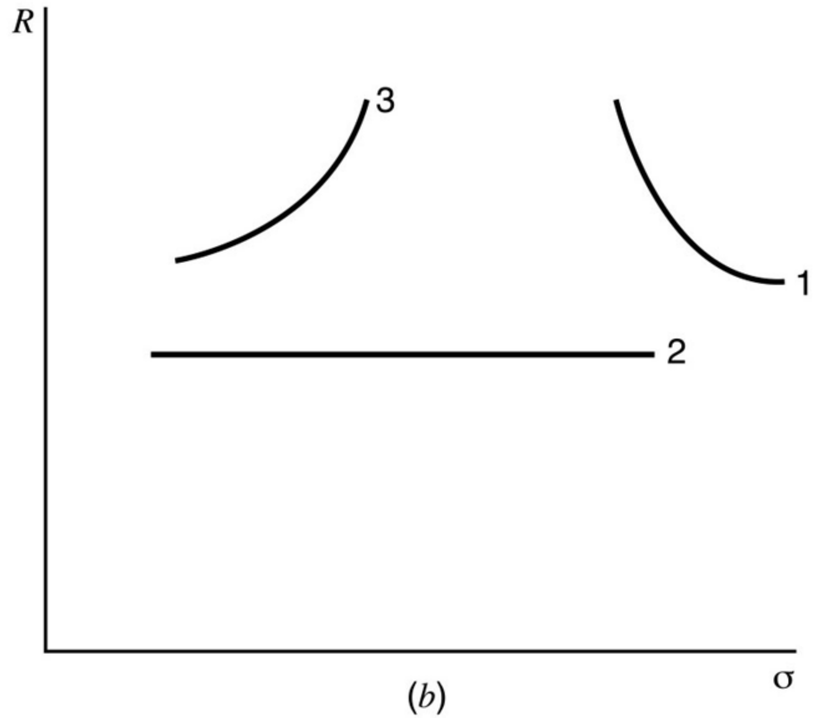

FIGURE 10-1 Characteristics of functions with different risk-aversion coefficients. (1) Utility function of a risk-seeking investor. (2) Utility function of a risk-neutral investor. (3) Utility function of a risk-averse investor.

Aversão absoluta ao risco

$$A(W) = \frac{-U''(W)}{U'(W)}$$

Table 10.7 Changes in Absolute Risk Aversion with Wealth

Condition	Definition	Property of $A(W)$[a]	Example[b]
Increasing absolute risk aversion	As wealth increases hold fewer dollars in risky assets	$A'(W) > 0$	$W - CW^2$
Constant absolute risk aversion	As wealth increases hold same dollar amount in risky assets	$A'(W) = 0$	$-e^{-CW}$
Decreasing absolute risk aversion	As wealth increases hold more dollars in risky assets	$A'(W) < 0$	$\ln W$

[a] $A'(W)$ is the first derivative of $A(W)$ with respect to wealth.
[b] The proof is left to the reader.

Aversão relativa ao risco

$$R(W) = \frac{-WU''(W)}{U'(W)} = WA(W)$$

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