Models in Finance - Class 8

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Wilkie model - The updating equations (cont.)

- The Equity dividend yield $Y(t)$:

$$Y(t) = \frac{D(t)}{P(t)},$$

where $D(t)$ is the dividend at time t and $P(t)$ is the share price at time t.

- Updating equation:

$$Y(t) = \exp\left\{YW.I(t) + \log(YMU) + YN(t)\right\},$$

where

$$YN(t) = YA.YN(t-1) + YE(t) \tag{1}$$

is the updating equation for the intermediary process $YN(t)$.

- Note that:

$$\log(Y(t)) = YW.I(t) + \log(YMU) + YN(t) \tag{2}$$

Wilkie model - dividend yield

- In the updating equations:
- $YE(t) = YSD.YZ(t)$.
- YA and YSD are parameters to be estimated.
- $YZ(t)$ is a series of i.i.d. standard normal r.v.
- The normal r.v. $YZ(t)$ are independent of the normal r.v. $QZ(t)$.
- If we replace $YN(t)$ given by (1) in (2):

$$\log Y(t) = YW.I(t) + \log(YMU) + YA.YN(t-1) + YSD.YZ(t),$$

but we still have the term $N(t-1)$, but from (2),
$\log Y(t-1) = YW.I(t-1) + \log(YMU) + YN(t-1)$ and therefore:

$$\begin{aligned}\log Y(t) = &\{YW.I(t) + \log(YMU)\} + \\ &+ YA.\left[\log Y(t-1) - \{YW.I(t-1) + \log(YMU)\}\right] + \\ &+ YSD.YZ(t).\end{aligned}$$

Wilkie model - dividend yield

- The structure of the updating equation for the equity dividend yield is similar to that of inflation $I(t)$, but with log for $Y(t)$ (reason: you cannot have negative values for dividend yield)
- this year's value = long-run mean $+YA \times$ (last year's value $-$ long-run mean) + "a shock to the system".
- Another difference: long-run mean $YW.I(t) + \log(YMU)$ is not constant and depends on the inflation $I(t)$. Economical reason? For investors, higher inflation should be compensated by higher yields.
- YSD determines the size of the "shock to the system".

Wilkie model - dividend income

Dividend income $D(t)$ updating eq.:

$$D(t) = D(t-1)\exp\{DW.DM(t) + DX.I(t) + DMU + \\ +DY.YE(t-1) + DB.DE(t-1) + DE(t)\},$$

where

- $YE(t) = YSD.YZ(t)$
- $DE(t) = DSD.DZ(t)$
- DW, DX, DY, DMU, DB, YSD and DSD are parameters to be estimated.
- $YZ(t)$ and $DZ(t)$ are series of iid standard normal r.v.
- The process $DM(t)$ is defined by:

$$DM(t) = DD.I(t) + (1-DD)DM(t-1).$$

The processes of this type are known as "exponentially-weighted moving average"processes. $DM(t)$ is a weighted average of the inflation that puts more weight on the most recent inflation figures.

Wilkie model - forece of dividend growth

- The force of dividend growth $K(t)$ updating eq.:

$$K(t) = DI(t) + DMU + DY.YE(t-1) + DB.DE(t-1) + DE(t).$$

- Note: $K(t)$ represents annual change in the log of dividend income:

$$K(t) = \log D(t) - \log D(t-1) = \log \frac{D(t)}{D(t-1)}.$$

- where $DI(t) = DW.DM(t) + DX.I(t)$ (this substitution could also be used in the dividend income eq.) .
- $YE(t)$, $DE(t)$ and all the parameters are the same as in the dividend income eq.
- $DZ(t)$ is a series of iid standard normal r.v.

Wilkie model- the real yield

- The real yield $R(t)$ updating eq. (real yield is the yield on a perpetual index-linked bond):

$$\log R(t) = \log(RMU) + RA\left[\log R(t-1) - \log(RMU)\right] + RBC.CE(t) + RE \quad (3)$$

where:

- $CE(t) = CSD.CZ(t)$
- $RE(t) = RSD.RZ(t)$
- RMU, RA, RBC, CSD and RSD are parameters to be estimated.
- $CZ(t)$ and $RZ(t)$ are series of iid standard normal r.v.

Wilkie model- the real yield

- Structure: this year's value = long-run mean $+RA \times$ (last year's value $-$ long-run mean) + "a shock to the system"+ another "shock to the system".
- Note that we have $\log R(t)$ and not $R(t)$ (no negative real yields) and we have two "shocks to the system".
- Exercise: Interpret the different terms in equation (3).
- The $CE(t) = CSD.CZ(t)$ is the "shock to the system"from the updating equation used to model conventional bond yields.
- $CZ(t)$ and $RZ(t)$ are not combined into a single white noise process because of the correlations that exist between conventional and index-linked bonds.

VARMA Representation

- The eqs. can be rep. as a "vector autor. moving average"(VARMA)

$$\begin{bmatrix} I(t) \\ K(t) \\ \log Y(t) \\ \log R(t) \end{bmatrix} = \begin{bmatrix} QMU \\ DD(DMU + QMU) \\ \log(YMU).(1 - YA) + YW.QMU(1 - YA) \\ \log(RMU) \end{bmatrix}$$

$$+ A \begin{bmatrix} I(t-1) - QMU \\ K(t-1) \\ \log Y(t-1) \\ \log R(t-1) - \log(RMU) \end{bmatrix} + B_0 \begin{bmatrix} QZ(t) \\ DZ(t) \\ YZ(t) \\ RZ(t) \\ CZ(t) \end{bmatrix}$$

$$+ B_1 \begin{bmatrix} QZ(t-1) \\ DZ(t-1) \\ YZ(t-1) \\ RZ(t-1) \\ CZ(t-1) \end{bmatrix} + B_2 \begin{bmatrix} QZ(t-2) \\ DZ(t-2) \\ YZ(t-2) \\ RZ(t-2) \\ CZ(t-2) \end{bmatrix}$$

VARMA Representation

- where $QZ(t)$, $DZ(t)$, $YZ(t)$, $RZ(t)$, $CZ(t)$ are independent, identically distributed $N(0,1)$ and all other non-time dependent parameters are fixed constants.
- $A =$
$$\begin{bmatrix} QA & 0 & 0 & 0 \\ (1 + DW.DD - DW)(QA - 1) + DD & 1 - DD & 0 & 0 \\ YW(QA - YA) & 0 & YA & 0 \\ 0 & 0 & 0 & RA \end{bmatrix},$$
- $B_0 =$
$$\begin{bmatrix} QSD & 0 & 0 & 0 & 0 \\ QSD(DW.DD + 1 - DW) & DSD & 0 & 0 & 0 \\ QSD & 0 & YSD & 0 & 0 \\ 0 & 0 & 0 & RSD & RBC.CSD \end{bmatrix}$$

VARMA Representation

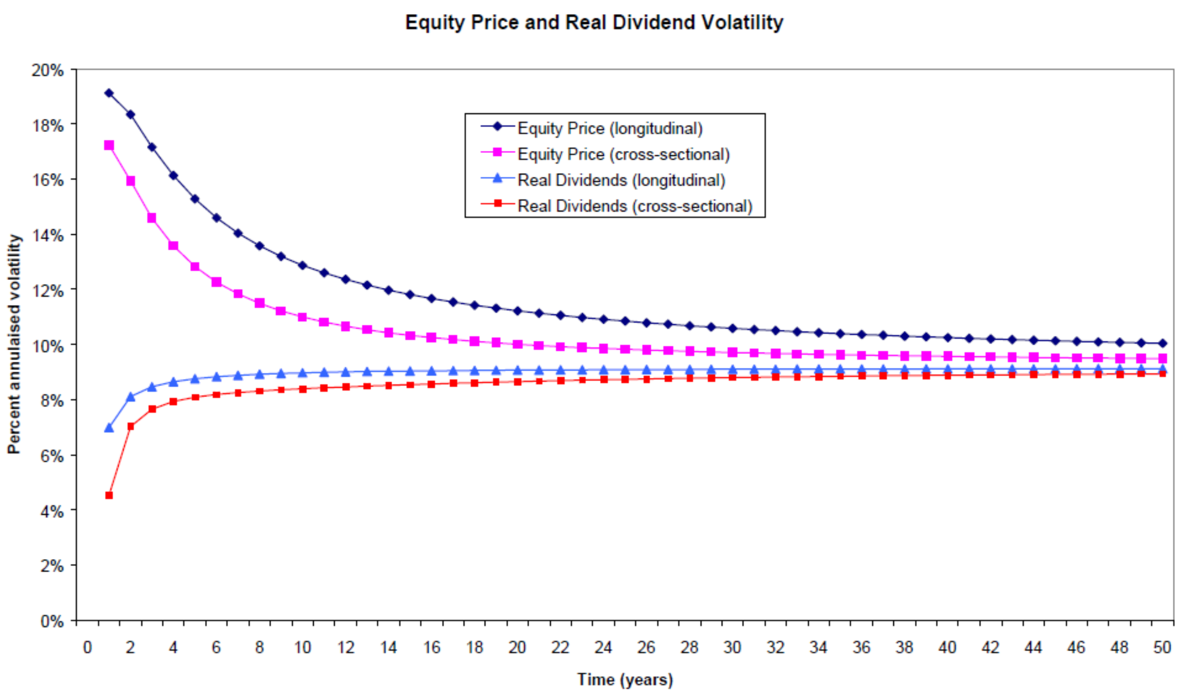
- where $QZ(t)$, $DZ(t)$, $YZ(t)$, $RZ(t)$, $CZ(t)$ are independent, identically distributed $N(0,1)$ and all other non-time dependent parameters are fixed constants.
- $B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & DSD.(DB+1-DD) & DY.YSD & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,
- $B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & DB.DSD\,(1-DD) & YD.YSD(1-DD) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

VARMA Representation

- Exercise: verify that these matrix equations give the same updating equation we had before for the inflation index, namely: $I(t) = QMU + QA\,[I\,(t-1) - QMU] + QE(t)$.
- Students will not be required to know the Wilkie model in particular or to reproduce these exact matrices in the examination but will be required to understand the principles behind how they are developed and be able to apply these principles.

Volatility analysis

- We can use the Wilkie model to demonstrate the differences between longitudinal and cross-sectional properties of time series models.
- Equity price index values, P(t), can be derived from Wilkie model by: $P(t) = \frac{D(t)}{Y(t)}$.
- The annualised cross-sectional and longitudinal volatility of the real dividend growth and the equity price in the Wilkie model can be calculated (chart below).
- Longitudinal volatilities are higher (it is expected, because longitudinal volatilities are unconditional values, while cross-sectional volatilities depend on the information set: more information, less volatility)



Volatility analysis

- Volatilities all converge to a common point because they are cointegrated.
- 2 processes with stationary increments are cointegrated if their movements are correlated in such a way that some linear combination of them is stationary.
- In the Wilkie model, the movements of equity prices and real dividends is essentially inflation and we modelled inflation as stationary process.
- The equity price volatility depends strongly on term (term structure): the downward slope evidences strong negative correlation. The real dividends are closer to a random walk (volatility varies little and depends weakly on term).

Total returns

- Total return is obtained if we combine the rate of growth in share price $P(t)$ with the income received from dividends: this is the figure the investors will be more interested in.
- Total return on an equity from t to $t+1$:

$$\frac{D\left(t+1\right)\left(1+\frac{1}{Y\left(t+1\right)}\right)}{D(t)/Y(t)} = Y\left(t\right)\left(1+\frac{1}{Y(t+1)}\right)\exp\left[K\left(t+1\right)\right].$$

- Total return on an index-linked bond:

$$\frac{Q\left(t+1\right)\left(1+\frac{1}{R\left(t+1\right)}\right)}{Q(t)/R(t)} = R\left(t\right)\left(1+\frac{1}{R(t+1)}\right)\exp\left[I\left(t+1\right)\right].$$

- In general, total return is $\frac{P(t+1)+D(t+1)}{P(t)}$.

Total returns

- Exercise: Derive the total return for the index-linked bond.
- Equity risk-premium can be defined as the conditional expectation of the log relative return on equities and index linked bonds (with low risk):

$$\log\left[\frac{Y(t)}{R(t)}\right] +$$
$$E\left[\log\left(1+\frac{1}{Y(t+1)}\right) - \log\left(1+\frac{1}{R(t+1)}\right) + K(t+1) - I(t+1)\right| U(t)$$

  where $U(t)$ is the state variable vector (with the values of all the economic variables at time t).
- The equity risk premium at time t is a function of the state variable $U(t)$.
- If the equity risk premium is high, then equities are expected to return more than index linked bonds (with low risk), while if the premium is low then index linked are expected to outperform in the next year.

Total returns

- Relation between longitudinal distribution of the equity risk premium and market efficiency?
- If markets were efficient, we would expect risk premiums to remain in a narrow range; otherwise excess profits could be earned by holding equities when the risk premium is high and holding index-linked bonds (low risk) otherwise.
- The wide variation in the equity risk premium in the Wilkie model $\Longrightarrow$ the equity and bond markets are not efficiently priced.
- Wilkie model is inconsistent with the weak form of the Efficient markets hypothesis.