Stochastic integration and Itô formula

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- The stochastic integral $I_t(F) = \int_0^t \int_E F(s,x) M(ds,dx)$, with $F \in \mathcal{H}_2$, satisfies:
  1. I_t is a linear operator
  2. $\mathbb{E}[I_t(F)] = 0$;
     $E\left[(I_t(F))^2\right] = \int_0^t \int_{E-\{0\}} \mathbb{E}\left[|F(s,x)|^2\right] \nu(dx)\, ds + \delta_0(E) \int_0^t \mathbb{E}\left[|F(s,0)|^2\right] ds.$
  3. $\{I_t(F), t \geq 0\}$ is $\{\mathcal{F}_t\}$ adapted
  4. $\{I_t(F),\ t \geq 0\}$ is a square-integrable martingale.

Sketch of the Proof of (3): Let $(F_n, n \in \mathbb{N})$ be a sequence of simple processes in $\mathcal{H}_2$ converging to F.
Then $(I_t(F_n), t \geq 0)$ is adapted and $I_t(F_n) \longrightarrow I_t(F)$ in L^2.
Therefore, there is a subsequence $(F_{n_k}; n_k \in \mathbb{N})$ such that $I_t(F_{n_k}) \longrightarrow I_t(F)$ a.s. as $n_k \to \infty$.

Therefore $\{I_t(F), \ t \geq 0\}$ is $\{\mathcal{F}_t\}$ adapted. ■

Sketch of the Proof of (4):
Let F be a simple process in $\mathcal{H}_2$ and choose $0 < s = t_l < t_{l+1} < t$.

Then $I_t(F) = I_s(F) + I_{s,t}(F)$ and by prop. (3),

$$\mathbb{E}_s(I_t(F)) = I_s(F) + \mathbb{E}_s(I_{s,t}(F))$$

Moreover,

$$\begin{aligned}\mathbb{E}_s(I_{s,t}(F)) &= \mathbb{E}_s\left(\sum_{j=l+1}^{m}\sum_{k=1}^{n} F_k(t_j) M((t_j, t_{j+1}], A_k)\right) \\ &= \sum_{j=l+1}^{m}\sum_{k=1}^{n} \mathbb{E}_s(F_k(t_j)) \mathbb{E}_s[M((t_j, t_{j+1}], A_k)] = 0.\end{aligned}$$

Therefore $\mathbb{E}_s(I_t(F)) = I_s(F)$ and $\{I_t(F), \ t \geq 0\}$ is a martingale.
Now, let $(F_n, n \in \mathbb{N})$ be a sequence of simple processes converging to F in L^2.
It can be proved that (see Applebaum) $\mathbb{E}_s(I_t(F_n)) \to \mathbb{E}_s(I_t(F))$ in L^2 and therefore $\mathbb{E}_s(I_t(F)) = I_s(F)$ is a square-integrable martingale■

- The stochastic integrals can be defined in an extended space: $\mathcal{P}_2(T,E)$ (where $\mathcal{H}_2 \subset \mathcal{P}_2(T,E)$), defined as the set of all mappings $F : [0,T] \times E \times \Omega \to \mathbb{R}$
  1) F is predictable
  2)

$$P\left[\int_0^T \int_{E-\{0\}} |F(t,x)|^2 \nu(dx)\, dt < \infty\right] = 1, \tag{1}$$

$$P\left[\int_0^T |F(t,0)|^2 dt < \infty\right] = 1. \tag{2}$$

- If $F \in \mathcal{P}_2(T,E)$ then $\{I_t(F),\ t \geq 0\}$ is a local martingale but not necessarily a martingale.
- If $E = \{0\}$ we use the notation $\mathcal{P}_2(T)$ for $\mathcal{P}_2(T,E)$.
- Therefore $\mathcal{P}_2(T)$ is the set of all predictable mappings $G : [0,T] \times \Omega \to \mathbb{R}$ such that

$$P\left[\int_0^T |G(t)|^2 dt < \infty\right] = 1$$

Lévy type stochastic integrals

- We say Y is a Lévy type stochastic integral if

$$Y_t^i = Y_0 + \int_0^t G^i(s)\, ds + \int_0^t F_j^i(s)\, dB_s^j + \int_0^t \int_{|x|<1} H^i(s,x)\, \widetilde{N}(ds,dx)$$
$$+ \int_0^t \int_{|x|\geq 1} K^i(s,x)\, N(ds,dx), \quad i = 1,...,d,\ j = 1,...,m \tag{3}$$

  where $\left|G^i\right|^{\frac{1}{2}}$, $F_j^i \in \mathcal{P}_2(T)$ and $H^i \in \mathcal{P}_2(T,E)$ and K is predictable.
- With stochastic differentials notation, in the one-dimensional case, we can write:

$$dY(t) = G(t)\, dt + F(t)\, dB(t) + \int_{|x|<1} H(t,x)\, \widetilde{N}(dt,dx)$$
$$+ \int_{|x|\geq 1} K(t,x)\, N(dt,dx).$$

Lévy type stochastic integrals

- Let M be an adapted and left-continuous process. Then we can define a new process $\{Z_t, t \geq 0\}$ by

$$dZ\,(t) = M\,(t)\,dY\,(t)$$

or

$$dZ(t) = M\,(t)\,G\,(t)\,dt + M\,(t)\,F\,(t)\,dB\,(t) + M\,(t)\,H\,(t,x)\,\widetilde{N}\,(dt,dx) + M\,(t)\,K\,(t,x)\,N\,(dt,dx)\,.$$

7

Example - Lévy stochastic integrals

- X: Lévy process with characteristics (b, A, ν) and Lévy-Itô decomposition

$$X\,(t) = bt + B_A\,(t) + \int_{|x|<1} x\widetilde{N}\,(t,dx) + \int_{|x|\geq 1} xN\,(t,dx)\,.$$

Let $L \in \mathcal{P}_2\,(t)$ for all $t \geq 0$. and choose in (3) $F_j^i = A_j^i L$, $H^i = K^i = x^i L$.

- The process Y such that

$$dY\,(t) = L\,(t)\,dX\,(t)$$

is called a Lévy stochastic integral.

8

Example - Ornstein Uhlenbeck (OU) process

- OU process:

$$Y(t) = e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)} dX(s),$$

  where y_0 is fixed.
- This process can be used for volatility modelling in finance.
- Exercise: Prove that if X is a one-dimensional Brownian motion then $Y(t)$ is a Gaussian process with mean $e^{-\lambda t} y_0$ and variance $\frac{1}{2\lambda}\left(1 - e^{-2\lambda t}\right)$

Example - Ornstein Uhlenbeck (OU) process

- In differential form the OU process is the solution of the SDE:

$$dY(t) = -\lambda Y(t)\, dt + dX(t),$$

  which is known as the Langevin equation (is a stochastic differential equation).
- The Langevin equation is also a model for the physical phenomenon of Brownian motion: includes the viscous drag of the medium on the particle as well as random fluctuations.

Itô formula for Poisson stochastic integrals

- Consider the Poisson stoch. integral $W(t) = W(0) + \int_0^t \int_A K(s,x) N(ds,dx)$, with A bounded below and K predictable.

Lemma

(Itô formula 1): If $f \in C(\mathbb{R})$ then

$$f(W(t)) - f(W(0)) = \int_0^t \int_A [f(W(s-) + K(s,x)) - f(W(s-))] N(ds,dx) \quad a.s.$$

Itô formula

Itô formula for Poisson stochastic integrals

Proof: Let $Y(t) = \int_A xN(t,dx)$. The jump times of Y can be defined by $T_0^A = 0$, $T_n^A = \inf\left\{t > T_{n-1}^A; \Delta Y(t) \in A\right\}$
Then

$$f(W(t)) - f(W(0)) = \sum_{0 \le s \le t} [f(W(s)) - f(W(s-))]$$

$$= \sum_{n=1}^{\infty} \left[f\left(W\left(t \wedge T_n^A\right)\right) - f\left(W\left(t \wedge T_{n-1}^A\right)\right)\right]$$

$$= \sum_{n=1}^{\infty} f\left(W\left(t \wedge T_n^A -\right) + K\left(t \wedge T_n^A, \Delta Y\left(t \wedge T_n^A\right)\right)\right) - f\left(W\left(t \wedge T_n^A -\right)\right)$$

$$= \int_0^t \int_A [f(W(s-) + K(s,x)) - f(W(s-))] N(ds,dx).$$

Itô formula for Brownian motion

- Let M be a Brownian integral with drift:

$$M^i(t) = \int_0^t F_j^i(s)\, dB^j(s) + \int_0^t G^i(s)\, ds,$$

with $F_j^i, \left|G^i\right|^{\frac{1}{2}} \in \mathcal{P}_2(t)$.

- Let us define the quadratic variation process:

$$\left[M^i, M^j\right](t) = \sum_{k=1}^{m} \int_0^t F_k^i(s)\, F_k^j(s)\, ds.$$

Itô formula

Itô formula for Brownian motion

Theorem

(Itô formula 2) If $f \in C^2\left(\mathbb{R}^d\right)$ *then*

$$f(M(t)) - f(M(0)) = \int_0^t \partial_i f(M(s))\, dM^i(s) + \frac{1}{2} \int_0^t \partial_i \partial_j f(M(s))\, d\left[M^i, M^j\right](s). \quad a.s.$$

Proof: See Applebaum

Itô formula for Lévy type stochastic integrals

- Let

$$dY(t) = G(t)\,dt + F(t)\,dB(t) + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx)$$

- $dY_c(t) := G(t)\,dt + F(t)\,dB(t)$
- $dY_d(t) := \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx)$

Itô formula for Lévy type stochastic integrals

Theorem

(Itô formula 3): If $f \in C^2(\mathbb{R}^d)$ then

$$f(Y(t)) - f(Y(0)) = \int_0^t \partial_i f(Y(s-))\,dY_c^i(s) + \frac{1}{2}\int_0^t \partial_i\partial_j f(Y(s-))\,d\left[Y_c^i, Y_c^j\right](s)$$
$$+ \int_0^t \int_{|x|\geq 1} \left[f(Y(s-)+K(s,x)) - f(Y(s-))\right] N(ds,dx)$$
$$+ \int_0^t \int_{|x|<1} \left[f(Y(s-)+H(s,x)) - f(Y(s-))\right] \widetilde{N}(ds,dx)$$
$$+ \int_0^t \int_{|x|<1} \left[f(Y(s-)+H(s,x)) - f(Y(s-)) - H^i(s,x)\,\partial_i f(Y(s-))\right] \nu(dx)\,ds$$

- Proof: see Applebaum

Itô formula for Lévy type stochastic integrals

Theorem

(Itô formula 4): If $f \in C^2\left(\mathbb{R}^d\right)$ then

$$f(Y(t)) - f(Y(0)) = \int_0^t \partial_i f(Y(s-))\, dY^i(s) + \frac{1}{2}\int_0^t \partial_i \partial_j f(Y(s-))\, d\left[Y_c^i, Y_c^j\right](s)$$
$$+ \sum_{0 \le s \le t} \left[f(Y(s)) - f(Y(s-)) - \Delta Y^i(s)\, \partial_i f(Y(s-))\right].$$

- Proof: see Applebaum

Itô formula for Lévy type stochastic integrals

- Quadratic variation process for Y:

$$\left[Y^i, Y^j\right](t) = \left[Y_c^i, Y_c^j\right](t) + \sum_{0 \le s \le t} \Delta Y^i(s)\, \Delta Y^j(s).$$

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$$\left[Y^i, Y^j\right](t) = \sum_{k=1}^m \int_0^t F_k^i(s)\, F_k^j(s)\, ds + \int_0^t \int_{|x|<1} H^i(s,x)\, H^j(s,x)\, \widetilde{N}(ds,dx) \tag{4}$$
$$+ \int_0^t \int_{|x|\ge 1} K^i(s,x)\, K^j(s,x)\, N(ds,dx).$$

Itô's product formula

Theorem

If Y^1 and Y^2 are real valued Lévy type stochastic integrals then

$$Y^1(t)\,Y^2(t) = Y^1(0)\,Y^2(0) + \int_0^t Y^1(s-)\,dY^2(s) + \int_0^t Y^2(s-)\,dY^1(s) + \left[Y^1, Y^2\right](t).$$

Proof Take $f(x_1, x_2) = x_1 x_2$ and apply Itô's formula 4:

$$Y^1(t)\,Y^2(t) - Y^1(0)\,Y^2(0) = \int_0^t Y^1(s-)\,dY^2(s)$$
$$+ \int_0^t Y^2(s-)\,dY^1(s) + \left[Y^1_c, Y^2_c\right](t)$$
$$+ \sum_{0\leq s\leq t} \left[Y^1(s)\,Y^2(s) - Y^1(s-)\,Y^2(s-) - \Delta Y^1(s)\,Y^2(s-) - \Delta Y^2(s)\,Y^1(s-)\right]$$

and the result follows.

Itô's product formula

- Product formula in differential form:

$$d\left(Y^1(t)\,Y^2(t)\right) = Y^1(t-)\,dY^2(t) + Y^2(t-)\,dY^1(t) + d\left[Y^1, Y^2\right](t).$$

- The Itô correction arises as the result of the following formal product relations (see (4)):

$$dB^i(t)\,dB^j(t) = \delta^{ij}dt,$$
$$N(dt, dx)\,N(dt, dy) = N(dt, dx)\,\delta(x-y),$$

all other products of differential vanish.

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