Models in Finance - Class 16

Master in Actuarial Science

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Continuous time models: preliminary concepts

- $(\Omega, \mathcal{F}, P)$: probability space where P is the real-world probability measure
- S_t (the price process of the risky asset) is adapted (measurable with respect) to the filtration $\mathcal{F}_t$ (given $\mathcal{F}_t$, we know the value of S_u for all $u \leq t$).
- Risk-free cash bond which has a value at time t of B_t.
- We will assume that the risk-free rate of interest is constant $\Longrightarrow B_t$ is deterministic and $B_t = B_0 e^{rt}$.
- Let $\mathcal{F}_t$ be the filtration generated by S_u $(0 \leq u \leq t)$.

Continuous time models: preliminary concepts

- Recall that the market is complete if for any contingent claim X there is a replicating strategy or portfolio (ϕ_t, ψ_t).
- Example of a complete market: the binomial model (we could replicate any derivative payment contingent on the history of the underlying asset price).
- Another example of a complete market is the continuous-time lognormal model for share prices:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t\right),$$

where Z_t is a standard Brownian motion.

Continuous time models: preliminary concepts

- Two measures P and Q which apply to the same sigma-algebra $\mathcal{F}$ are said to be equivalent if for any event $E \in \mathcal{F} : P(E) > 0$ if and only if $Q(E) > 0$, where $P(E)$ and $Q(E)$ are the probabilities of E under P and Q respectively.
- For the binomial model, for the equivalence of P and Q the only constraint on the real-world measure P is that at any point in the binomial tree the probability of an up move lies strictly between 0 and 1. The only constraint on Q is the same.

Continuous time models: preliminary concepts

- Suppose that Z_t is a standard Brownian motion under P and let $X_t = \gamma t + \sigma Z_t$ be a Brownian motion with drift under P.
- Is there a measure Q under which X_t is a standard Brownian motion and which is equivalent to P ?
- Yes if $\sigma = 1$ but no if $\sigma \neq 1$
- In other words: we can change the drift of the Brownian motion but not the volatility.
- Theorem (Cameron-Martin-Girsanov): Suppose that Z_t is a standard Brownian motion under P and that γ_t is a previsible process. Then there exists a measure Q equivalent to P and where $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q. Conversely, if Z_t is a standard Brownian motion under P and if Q is equivalent to P then there exists a previsible process γ_t such that $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds$ is a standard Brownian motion under Q.

Continuous time models: preliminary concepts

- Assume that under P (geometric Bm):$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma Z_t\right)$. Then ($e^{-rt} S_t$ is the discounted price):
$$E_P\left[e^{-rt} S_t\right] = e^{(\mu - r)t}$$
and $e^{-rt} S_t$ is not a martingale under P (unless $\mu = r$).
- Take $\gamma_t = \gamma = \frac{\mu - r}{\sigma}$ and define $\widetilde{Z}_t = Z_t + \int_0^t \gamma_s ds = Z_t + \frac{(\mu - r)}{\sigma} t$. Then:
$$\begin{aligned} S_t &= S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{Z}_t - (\mu - r)\, t\right) \\ &= S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma \widetilde{Z}_t\right). \end{aligned}$$
By the Cameron-Martin-Girsanov theorem, exists Q equivalent to P such that $\widetilde{Z}_t$ is a Q-standard Bm.

Continuous time models: preliminary concepts

- And clearly, we have (for $u < t$):

$$E_Q\left[e^{-rt}S_t|\mathcal{F}_u\right] =$$
$$= e^{-rt}S_u E_Q\left[\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(t-u)+\sigma\left(\widetilde{Z}_t-\widetilde{Z}_u\right)\right)\right]$$
$$= e^{-ru}S_u E_Q\left[\exp\left(\left(-\frac{1}{2}\sigma^2\right)(t-u)+\sigma\left(\widetilde{Z}_t-\widetilde{Z}_u\right)\right)\right]$$
$$= e^{-ru}S_u e^{\left(-\frac{1}{2}\sigma^2\right)(t-u)+\frac{1}{2}\sigma^2(t-u)} = e^{-ru}S_u$$

- Therefore, the discounted price $e^{-rt}S_t$ is a Q-martingale.

Continuous time models: preliminary concepts

- Suppose that X_t is a P-martingale and Y_t is another P-martingale.
- Martingale Representation Theorem (MRT): Exists a unique previsible process ϕ_t such that

$$Y_t = Y_0 + \int_0^t \phi_s dX_s \quad \text{(or: } dY_t = \phi_t dX_t\text{)}$$

if and only if there is no other measure equivalent to P under which X_t is a martingale.

MRT for the binomial model

- MRT applied to the binomial model: Under Q:

$$X_{t+1} = \begin{cases} X_t + u(t, X_t) & \text{with probab. } q \\ X_t + d(t, X_t) & \text{with probab. } 1-q \end{cases}.$$

If it is a Q-martingale, then $q = \frac{-d}{u-d}$ (this uniquely specifies Q).

- If Y_t is also a martingale with respect to Q then Y_t must also follow a binomial model with:

$$Y_{t+1} = \begin{cases} Y_t + \widetilde{u}(t, Y_t) & \text{with probab. } q \\ Y_t + \widetilde{d}(t, Y_t) & \text{with probab. } 1-q \end{cases}$$

and $q = \frac{-\widetilde{d}}{\widetilde{u}-\widetilde{d}}$ and $\widetilde{d}(t, Y_t) = -\frac{q\widetilde{u}(t,Y_t)}{1-q}$. Then, if $\phi_{t+1} = \frac{\widetilde{u}(t,Y_t)}{u(t,X_t)}$ (ϕ_t is previsible, i.e. it is $\mathcal{F}_{t-1}$ measurable or $\mathcal{F}_{t-}$ measurable) we have the MRT:

$$Y_t = Y_0 + \sum_{s=1}^{t} \phi_s \Delta X_s, \qquad (1)$$

where $\Delta X_s = X_s - X_{s-1}$. Eq. (1) is equivalent to: $\Delta Y_t = \phi_t \Delta X_t$.

The Binomial model again:

- 5 steps which can be used to solve the problems of pricing and hedging of derivatives:

1. Establish the equivalent measure Q under which the discounted asset price process $D_t = e^{-rt} S_t$ is a martingale.
2. Define the fair price of the derivative: $V_t = e^{-r(n-t)} E_Q[C_n | \mathcal{F}_t]$ where C_n is the derivative payoff at time n.
3. Let $F_t = e^{-rt} V_t = e^{-rn} E_Q[C_n | \mathcal{F}_t]$. Under Q , F_t is a martingale.
4. The measure Q is the unique martingale measure. Therefore, by the MRT exists a previsible process ϕ_t (i.e. ϕ_t is $\mathcal{F}_{t-1}$ measurable) such that:

$$\Delta F_t = F_t - F_{t-1} = \phi_t (D_t - D_{t-1}) = \phi_t \Delta D_t.$$

The Binomial model again:

- We can calculate (see the core reading)

$$\phi_t = \frac{V_t(2j-1) - V_t(2j)}{S_{t-1}(j)(u_{t-1}(j) - d_{t-1}(j))}.$$

5. Let $\psi_t = F_t - \phi_t D_t$.
   Between times $t-1$ and t^- we hold the portfolio

$$\begin{cases} \phi_t \text{ units of asset } S_t \\ \psi_t \text{ of cash account } B_t. \end{cases}$$

   We can show that this porfolio is a self-financing portfolio and the value of the portfolio at time t will be V_t (see core reading): this hedging portfolio is a replicating portfolio and this implies that $V_t = e^{-r(n-t)} E_Q[C_n | \mathcal{F}_t]$ is indeed the fair price of the derivative at time t.

5 step method

1. Establish the equivalent martingale measure Q.
2. Propose a fair price for the derivative V_t and its discounted value $F_t = e^{-rt} V_t$.
3. Use the MRT to construct a hedging strategy (portfolio) (ϕ_t, ψ_t).
4. Show that the hedging strategy (ϕ_t, ψ_t) replicates the derivative payoff at time n.
5. Therefore V_t is the fair price of the derivative at time t.