Option pricing in exponential Lévy models with Partial integro-Differential Equations

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Outline

Diffusion dynamics

- Risk neutral dynamics described by a diffusion process

$$\frac{dS_t}{S_t} = rdt + \sigma(t, S_t)\, dW_t.$$

- The value $C(S,t)$ of European or Barrier option is the solution of the parabolic PDE (Black-Scholes PDE):

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2(t,S)\, S^2}{2}\frac{\partial^2 C}{\partial S^2} - rC(t,S) = 0,$$

with boundary conditions depending on the payoff of the option.

Lévy dynamics

- Risk neutral dynamics described by a Lévy process

$$S_t = S_0 \exp(rt + X_t), \tag{1}$$

where X is a Lévy process with triplet (γ, σ^2, ν) under some risk neutral measure $\mathbb{Q}$ such that $\widehat{S}_t = e^{-rt} S_t = \exp(X_t)$ is a martingale.

- $\widehat{S}_t = e^{-rt} S_t = \exp(X_t)$ is a martingale with respect to $\mathbb{Q}$. We know from a previous lecture (see lecture 7) that this is equivalent to

$$\gamma = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} \left(e^y - 1 - y\mathbf{1}_{\{|y|\leq 1\}}\right) \nu(dy). \tag{2}$$

- We will assume that (condition equivalent to the existence of 2nd moment for S_t)

$$\int_{|y|\geq 1} e^{2y} \nu(dy) < \infty.$$

- The value $C(S,t)$ of an European or Barrier option is the solution of the 2nd order partial integro-differential eq. (P.I.D.E.)

$$\frac{\partial C}{\partial t}+rS\frac{\partial C}{\partial S}+\frac{\sigma^2S^2}{2}\frac{\partial^2 C}{\partial S^2}-rC(t,S)$$
$$+\int_{\mathbb{R}}\left[C(t,Se^y)-C(t,S)-S(e^y-1)\frac{\partial C}{\partial S}(t,S)\right]\nu(dy)=0,$$

with boundary conditions depending on the payoff of the option.

- A Lévy process is a Markov processand the associated semigroup has infinitesimal generator $L:f\to Lf$ given by the integro-differential operator (for $f\in C^2(\mathbb{R})$ with compact support)

$$\begin{aligned} Lf(x) &= \lim_{t\to 0}\frac{E[f(x+X_t)]-f(x)}{t} \\ &= \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}+\gamma\frac{\partial f}{\partial x}+ \\ &+\int_{\mathbb{R}}\left[f(x+y)-f(x)-y\mathbf{1}_{\{|y|\le 1\}}\frac{\partial f}{\partial x}(x)\right]\nu(dy), \end{aligned} \tag{3}$$

and the transition or evolution operator is

$$P_tf(x)=E[f(x+X_t)].$$

- Replacing (2) in (3), we obtain

$$Lf(x)=\frac{\sigma^2}{2}\left[\frac{\partial^2 f}{\partial x^2}-\frac{\partial f}{\partial x}\right]+$$
$$+\int_{\mathbb{R}}\left[f(x+y)-f(x)-(e^y-1)\frac{\partial f}{\partial x}(x)\right]\nu(dy)$$

- The risk neutral dynamics SDE (under $\mathbb{Q}$) that corresponds to (1) is

$$\frac{d\widehat{S}_t}{\widehat{S}_t}=\sigma dW_t+\int_{\mathbb{R}}(e^x-1)\widetilde{N}(dt.dx). \tag{4}$$

- Value of European option with payoff $H(S_T)$ is $C_t = C(t,S)$ with

$$C(t,S) = E\left[e^{-r(T-t)} H(S_T) \,|S_t = S\right]$$

- Introducing the change of variable: $\tau = T - t$, $x = \ln(S/S_0)$ and defining $h(x) = H(S_0 e^x)$ and $f(\tau, x) = e^{r\tau} C(T - \tau, S_0 e^x)$, we get:

$$f(\tau, x) = E\left[h(x + r\tau + X_\tau)\right]. \tag{5}$$

- If h is in the domain of L then we can differentiate with respect to τ and we obtain (using the definition of the infinitesimal generator):

$$\begin{aligned} \frac{\partial f}{\partial \tau} &= Lf + r\frac{\partial f}{\partial x} \quad \text{on } (0,T] \times \mathbb{R}, \\ f(0,x) &= h(x). \end{aligned} \tag{6}$$

which is a Partial integro-differential equation (P.I.D.E.).

- Similarly, if f is smooth, then using a change of variable we obtain a similar equation:

$$\begin{aligned} &\frac{\partial C}{\partial t} + L^S C(t,S) - rC(t,S) = 0, \\ &C(T,S) = H(S), \end{aligned}$$

where

$$\begin{aligned} L^S f(x) = {} & rx\frac{\partial f}{\partial x} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2} + \\ & + \int_{\mathbb{R}} \left[f(xe^y) - f(x) - x(e^y - 1)\frac{\partial f}{\partial x}(x)\right] \nu(dy). \end{aligned}$$

is the infinitesimal generator of S_t (with state space $(0,\infty)$).

- This reasoning is heuristic: payoff is usually not in the domain of L and is usually not even differentiable. Example $h(x) = [K - S_0 e^x]^+$ for a put option.

- Assume that the Payoff H is Lipschitz: $|H(x) - H(y)| \leq c\,|x - y|$.
- Then $C_t = C(t, S)$ with $C(t, S) = E\left[e^{-r(T-t)} H(S_T)\,|S_t = S\right] = e^{-r(T-t)} E\left[H\left(Se^{r(T-t)+X_{T-t}}\right)\right].$

Proposition

If $\sigma > 0$ *then*

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} - rC(t, S)$$
$$+ \int_{\mathbb{R}}\left[C(t, Se^y) - C(t, S) - S(e^y - 1)\frac{\partial C}{\partial S}(t, S)\right]\nu(dy) = 0,$$
$$C(T, S) = H(S), \text{ for all } S > 0.$$

Proof.

idea: Apply Itô's formula do the martingale $\widehat{C}(t, S_t) = e^{-rt}C(t, S_t)$, identify the drift part and set it to zero. $C(t, S)$ is a smooth function of S (see Cont - [1]). Applying Itô's formula and using (4), we obtain:

$$d\widehat{C}_t = a(t)\,dt + dM_t,$$

where

$$a(t) = e^{-rt}\left[-rC + \frac{\partial C}{\partial t} + \frac{\sigma^2 S_{t-}^2}{2}\frac{\partial^2 C}{\partial S^2} + rS_{t-}\frac{\partial C}{\partial S}\right](t, S_{t-})$$
$$+ e^{-rt}\int_{\mathbb{R}}\left[C(t, S_{t-}e^x) - C(t, S_{t-}) - S_{t-}(e^x - 1)\frac{\partial C}{\partial S}(t, S_{t-})\right]\nu(dx),$$
$$dM_t = e^{-rt}\left[\frac{\partial C}{\partial S}(t, S_{t-})\,\sigma S_{t-}dW_t + \int_{\mathbb{R}}\left(C(t, S_{t-}e^x) - C(t, S_{t-})\right)\widetilde{N}(dt, dx)\right].$$

□

Proof.

(cont.) Let us now show that M is a martingale. Since H is Lipschitz and e^{X_t} is a martingale, we have that C is also Lipschitz

$$|C(t,S_1) - C(t,S_2)| \leq e^{-r(T-t)} \left| E\left[H\left(S_1 e^{r(T-t)+X_{T-t}}\right) - H\left(S_2 e^{r(T-t)+X_{T-t}}\right)\right]\right|$$
$$\leq c\,|S_1 - S_2|\, E\left[e^{X_{T-t}}\right] = c\,|S_1 - S_2|\,.$$

Therefore the predictable function $\psi(t,x) = C(t, S_{t-}e^x) - C(t,S_{t-})$ satisfies

$$E\left[\int_0^T \int |\psi(t,x)|^2 \nu(dx)\,dt\right] \leq$$
$$\leq c^2 \int (e^x - 1)^2 \nu(dx)\, E\left[\int_0^T S_{t-}^2 dt\right] < \infty.$$

Hence, $\int_0^t \int_{\mathbb{R}} e^{-rs}\left(C(s,S_{s-}e^x) - C(s,S_{s-})\right)\widetilde{N}(ds,dx)$ is a square-integrable martingale. □

Proof.

(cont.) Moreover, since C is Lipschitz, $\frac{\partial C}{\partial S}$ is bounded by a constant c.Therefore

$$E\left[\int_0^T \left(\frac{\partial C}{\partial S}(t,S_{t-})\,S_{t-}\right)^2 dt\right] \leq c^2 E\left[\int_0^T S_{t-}^2 dt\right] < \infty,$$

and $\int_0^t \int_{\mathbb{R}} e^{-rs}\frac{\partial C}{\partial S}(s,S_{s-})\,\sigma S_{s-} dW_s$ is also a square integrable martingale. Therefore $\widehat{C}_t - M_t = \int_0^t a(s)\,ds$ is a square integrable martingale.But $\int_0^t a(s)\,ds$ is also a continuous process with finite variation and therefore, $a(t) = 0$ a.s. in the $\mathbb{Q}$ measure. ■

□

- It is possible to prove that a sufficient condition to apply the previous Proposition, in the case of pure jump processes ($\sigma = 0$), is that

$$\exists \beta \in (0,2) : \liminf_{\epsilon \to 0} \epsilon^{-\beta} \int_{-\epsilon}^{\epsilon} |x|^2 \nu(dx) > 0. \quad (7)$$

- Condition (7) holds for Lévy densities behaving near zero as $\nu(x) = c/x^{1+\beta}$, with $\beta > 0$.
- The pure jump variance-Gamma model does not satisfies this condition. In this case, the P.I.D.E. reduces to a first order eq. but even the C^1 smoothness may fail.
- If $\sigma \neq 0$, one can use the P.I.D.E. in order to compute the option price.
- In pure jump models, if condition (7) falis, the smoothness of the option price with respect to the underlying may fail.

Feynman-Kac formula

- A Feynman-Kac formula is the following one (let $X_s^{t,x}$ denote the Lévy process at time $s > t$ such that $X_t^{t,x} = x$)

Proposition

Consider a bounded function $h \in L^\infty(\mathbb{R})$ and $\sigma > 0$. Then the Cauchy problem

$$\frac{\partial f}{\partial t}(t,x) + \gamma\frac{\partial f}{\partial x}(t,x) + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(t,x)$$
$$+\int\left[f(t,x+y) - f(t,x) - y\mathbf{1}_{|y|\leq 1}\frac{\partial f}{\partial x}(t,x)\right]\nu(dy) = 0,$$
$$f(T,x) = h(x), \text{ for all } x \in \mathbb{R}.$$

has a unique solution given by

$$f(t,x) = E\left[h\left(X_T^{t,x}\right)\right].$$

Example of lack of smoothness in pure jump models

- Variance Gamma process: the characteristic function of X_t is

$$\Phi_t(u) = \left(1 + \frac{u^2\sigma^2 k}{2} - i\theta k u\right)^{-\frac{t}{k}},$$

where

$$\nu(x) = \frac{1}{k|x|} e^{Ax - B|x|}.$$

In this case, the value of an European binary option with payoff $h(x) = \mathbf{1}_{x \geq x_0}$ is continuous but not differentiable in x for $t < \frac{k}{2}$: the option price has a vertical tangent at the money (see Cont - [2])

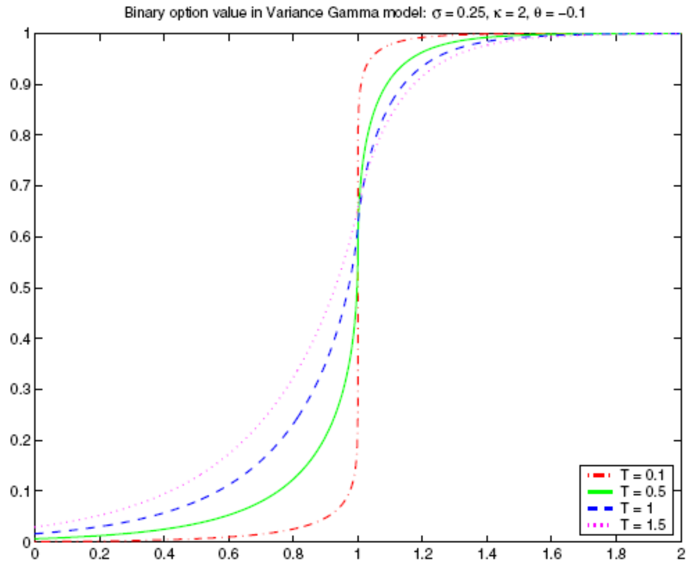

Fig. 1. Value of a binary option in the Variance Gamma model, as a function of the underlying

Viscosity solutions

- The notion of viscosity solution is an intrinsic definition of solution and does not impose a priori the existence of derivatives (continuity is enough).
- In the case of pure jump processes ($\sigma = 0$) that do not satisfy condition (7) like the Variance-Gamma process, one cans study the solution of the P.I.D.E., considering that these solutions are not necessarily classical solutions (with first and second derivatives well defined), but are viscosity solutions (only required to be continuous). This is discussed in detail in [1] and [2].

Numerical methods for P.I.D.E.'s

- Multinomial trees
- Finite difference methods
- Finite elements
- Galerkin methods

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