Models in Finance - Class 22

Master in Actuarial Science

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ISEG

Risk-neutral measure as a computational tool

- What is the effect on $r(t)$ of the transformation of P to Q?

$$
\begin{aligned}
dr(t) &= a(t, r(t))dt + b(t, r(t))dW_t \quad \text{under } P && (1)\\
&= a(t, r(t))dt + b(t, r(t)) \left(d\widetilde{W}_t - \gamma(t)\, dt \right)\\
&= \left(a(t, r(t)) - \gamma(t)\, b(t, r(t)) \right) dt + b(t, r(t)) d\widetilde{W}_t\\
&= \widetilde{a}(t, r(t))dt + b(t, r(t)) d\widetilde{W}_t, \quad \text{under } Q. && (2)
\end{aligned}
$$

where $\widetilde{a}(t, r(t)) = a(t, r(t)) - \gamma(t)\, b(t, r(t))$.

Risk-neutral measure as a computational tool

- Risk-neutral pricing formula:

$$B(t,T) = E_Q\left[\exp\left(-\int_t^T r(u)du\right)\bigg| r(t)\right]. \tag{3}$$

- Q is an artificial computational tool. It is determined by combining (a) the model for $r(t)$ under the real world measure P and (b) the market price of risk established from knowledge of the dynamics of one bond.
- When modellers use this approach to pricing, from the practical point of view they normally start by specifying the dynamics of $r(t)$ under Q in order to calculate bond prices. Second, they specify the market price of risk as a component of the model, and this allows us to determine the dynamics of $r(t)$ under P.

Models for the term structure of interest rates

- Several models are based on the short-rate $r(t)$ in the risk-neutral framework: for example, the Vasicek and Cox-Ingersoll-Ross (CIR) models.
- These two models are time homogeneous: that is, the future dynamics of $r(t)$ only depend upon the current value of $r(t)$ rather than what the present time t actually is.

Vasicek model

- The dynamics of the Vasicek model under Q is:

$$dr(t) = \alpha \left(\mu - r(t)\right) dt + \sigma d\widetilde{W}_t, \tag{4}$$

where $\widetilde{W}_t$ is a standard Bm under Q, and the parameter α is positive.
- Note that the Vasicek model SDE is the same as for the Ornstein-Uhlenbeck process with mean-reversion.
- As we have deduced before in the first chapters (see the discussion of the Ornstein-Uhlenbeck processes), the solution of this SDE is:

$$r(t) = r(0)e^{-\alpha t} + \mu \left(1 - e^{-\alpha t}\right) + \sigma \int_0^t e^{-\alpha(t-u)} d\widetilde{W}_u. \tag{5}$$

Vasicek model

- In the Vasicek model, we can deduce the following formula for the bond prices:

$$B(t,T) = e^{a(\tau) - b(\tau) r(t)}, \tag{6}$$

where

$$\tau = T - t,$$
$$b(\tau) = \frac{1 - e^{-\alpha\tau}}{\alpha},$$
$$a(\tau) = \left(b(\tau) - \tau\right) \left[\mu - \frac{\sigma^2}{2\alpha^2}\right] - \frac{\sigma^2}{4\alpha} b(\tau)^2.$$

Vasicek model

- Exercise: Show that the instantaneous forward rate for the Vasicek model can be expressed as:

$$f(t,T) = r(t)e^{-\alpha\tau} + \left[\mu - \frac{\sigma^2}{2\alpha^2}\right]\left(1 - e^{-\alpha\tau}\right) + \frac{\sigma^2}{2\alpha^2}\left(e^{-\alpha\tau} - e^{-2\alpha\tau}\right).$$

- Given that the current time is t, we can show that $r(T)$ has normal distribution.
- We can use the bivariate normality of both $r(T)$ and $\int_t^T r(u)du$ in order to deduce simple formulae for the prices of European options on both zero-coupon and coupon-bearing bonds.

Vasicek model

- Example: How can we derive a formula for the value of a call option on a bond (with maturity T) that gives the holder the option to buy the bond at a specified time s (where $t < s < T$) by the price K?
- Solution: The payoff at time s is $\max\left[B(s,T) - K, 0\right]$. The Risk-neutral pricing formula is

$$V_t = E_Q\left[\exp\left(-\int_t^s r(u)du\right) \times \max\left[B(s,T) - K, 0\right] \middle| \mathcal{F}_t\right]. \quad (7)$$

- Option prices for zero-coupon bonds closely resemble the Black-Scholes formula for equity option prices.

Vasicek model

- Main drawback of the Vasicek model: interest rates can go negative.
- If the probability of negative rates is small possibly because the time horizon is short, then this is not a problem.
- In other cases (especially longer-term actuarial applications) the probability and severity of negative interest rates can be significant and this is a serious problem.