

6.1 difference equations

6.1. Solve, through successive iteration, the following 1st order difference equations.

a) $y_{t+1} = \alpha y_t, y_0 = \beta$

b) $y_t = \alpha y_{t-1} + a, y_0 = \beta$

c) $y_{t+n} = y_{t+n-1} - a, y_1 = y_0 (n \in \mathbb{N})$ d) $\begin{cases} y_t = \alpha y_{t-1} + a, & t \leq t^* \\ y_t = \beta y_{t-1} + b, & t > t^* \end{cases} \quad y_0 = C$

6.2. Solve the following first order homogeneous difference equations.

a) $y_t = \alpha y_{t-1}, y_0 = 10$ b) $\Delta y_t = y_t(1 - \alpha), y_0 = \beta$

c) $y_{t+1} + 3y_t = 0, y_0 = 4$ d) $2y_{t+2} - y_{t+1} = 0, y_0 = 7$

e) $y_{t-2} = 0.2y_{t-3}, y_0 = 1$

Solution:

a) $y_t = 10\alpha^t$; b) $y_t = \beta(2 - \alpha)^t$; c) $y_t = 4(-3)^t$; d) $y_t = 7(\frac{1}{2})^t$; e) $y_t = (0.2)^t$.

6.3. Solve the following first order difference equations.

a) $y_t = \alpha y_{t-1} + 2$ b) $\Delta y_t = y_t(1 - \alpha) + 2.4^t$

c) $y_{t+1} + 3y_t = 4t^2$ d) $y_{t+1} = y_t + 2t3^t$

e) $y_{t-2} = 3y_{t-3} + 8t + 4^t$ f) $\Delta y_t = y_t(1 - \alpha) + \alpha - 1$

g) $y_t = 2y_{t-1} + 2^t$

Solution:

a) if $\alpha \neq 1, y_t = C\alpha^t + \frac{2}{1-\alpha}$; if $\alpha = 1, y_t = C + 2t$ b) If $\alpha \neq -2, y_t = C(2 - \alpha)^t + \frac{2}{2+\alpha}4^t$; if $\alpha = -2, y_t = C4^t + \frac{1}{2}t4^t$ c) $y_t = C(-3)^t + t^2 - \frac{1}{2}t - \frac{1}{8}$; d) $y_t = C + (t - \frac{3}{2})3^t$; e) $y_t = C3^t - 14 - 4t + 4^{t+3}$; f) If $\alpha \neq 1, y_t = C(2 - \alpha)^t + 1$; if $\alpha = 1, y_t = C(2 - \alpha)^t$ g) $y_t = C2^t + t2^t$.

6.4. Solve the following second order homogeneous difference equations.

a) $y_t + \frac{1}{2}y_{t-1} - \frac{1}{2}y_{t-2} = 0$, sabendo que a solução passa nos pontos (0,1) e (1,1)

b) $y_{t+2} - 4y_{t+1} + 4y_t = 0$, $y(0) = 1, y(1) = 2$; c) $y_{t+2} - y_{t+1} + \frac{1}{2}y_t = 0$

d) $\Delta^2 y_t = 0$

Solution:

a) $y_t = -\frac{1}{3}(-1)^t + \frac{4}{3}(\frac{1}{2})^t$; b) $y_t = 2^t$; c) $y_t = (\frac{\sqrt{2}}{2})^t(C_1 \cos \frac{\pi}{4}t + C_2 \sin \frac{\pi}{4}t)$, d) $y_t = C_1 + C_2t$.

6.5. Solve the following second order difference equations.

a) $y_t + \frac{1}{2}y_{t-1} - \frac{1}{2}y_{t-2} = 1$ b) $y_{t+2} - 4y_{t+1} + 4y_t = 2^t$ c) $y_{t+2} - 4y_{t+1} + 4y_t = t$

d) $y_{t+2} - 4y_{t+1} + 4y_t = t + 2^t$ e) $y_{t+2} - y_{t+1} + \frac{1}{2}y_t = t^2$ f) $\Delta^2 y_t = t + e^t$

Solution:

a) $y_t = C_1(\frac{1}{2})^t + C_2(-1)^t + 1$; b) $y_t = (C_1 + C_2t)2^t + t^2 2^{t-3}$; c) $y_t = (C_1 + C_2t)2^t + t + 2$ d) $y_t = (C_1 + C_2t)2^t + t^2 2^{t-3} + t + 2$ e) $y_t = (\frac{\sqrt{2}}{2})^t (C_1 \cos \frac{\pi}{4}t + C_2 \sin \frac{\pi}{4}t) + 2t^2 - 8t + 4$; f) $y_t = C_1 + C_2t + \frac{t^3}{6} - \frac{t^2}{2} + \frac{1}{e^2 - 2e + 1}e^t$.

6.6. Solve the following difference equations.

a) $2y_t - y_{t-1} - y_{t-2} = 0$ b) $y_{t+5} - 4y_{t+4} + 4y_{t+3} = 12(3^t)$ c) $2y_{t+1} - 3y_t + y_{t-1} = 3$

d) $y_{t+3} - 2y_{t+2} + y_{t+1} = 4$ e) $3y_{t+2} + y_{t+1} - 2y_t = 5(-1)^t$

Solution: a) $y_t = C_1 + C_2(-\frac{1}{2})^t$, b) $y_t = (C_1 + C_2t)2^t + 4(3^{t-2})$, c) $y_t = C_1 + C_2(\frac{1}{2})^t + 3t$, d) $y_t = C_1 + C_2t + 2t^2$, e) $y_t = C_1(-1)^t + C_2(\frac{2}{3})^t + t(-1)^t$