EXERCISES FOR CHAPTER TWO

1. A box contains balls of three different colors: 5 black, 3 green and 7 red. A random experiment consists of taking out two balls without replacement. Black balls worth 1 point, green balls 2 points and red balls 3 points. Let X be the random variable that represents the sum of points scored.

    a. Write the reverse image of $[3,5) \subset \mathcal{R}$.
    b. Determine the cumulative distribution function of random variable X.
    c. Evaluate $P(X > 3 | X < 6)$.
    d. Classify the random variable X.

2. Let $\Omega = \{(x,y): x^2 + y^2 \leq 1\}$ be the sample space of a random experiment. In this space the probability associated with any event $A \subset S$ is proportional to the area of the figure corresponding to event A. Let $U[(x,y)]$ be the random variable defined by the distance of (x,y) from origin $(0,0)$.

    a. Determine the cumulative distribution function of random variable U.
    b. Evaluate $(0.2 < U < 0.5)$,$P(U > 0.5 | U > 0.2)$.
    a. Classify the random variable U.

3. Let $F_X(x) = \begin{cases} 0 & (x < 1) \\ x/3 & (1 \leq x < 2) \\ 1 & (x \geq 2) \end{cases}$ be the cumulative distribution function of random variable X.

    a. Verify that it is a cumulative distribution function of random variable X.
    b. Classify the random variable X.

4. Consider the random variable X with cumulative distribution function given by:

$$F_X(x) = \begin{cases} 0 & (x < 0) \\ 1 - (2/3)e^{-x} & (x \geq 0) \end{cases}$$

    a. Calculate $P(X > 0)$.
    b. Graph the above function and classify the random variable X.

5. A box has n white balls (W) and just one black ball (B). Balls are taken out without replacement until the black ball comes out. Let X be the random variable representing the number of extractions done.

    a. What is the image of event $A = \{B, WB, WWB, WWWB\}$?
    b. What is the reverse image of $(0,5) \subset \mathcal{R}$?
    c. Classify the random variable and get its cumulative distribution function.

6. Let X be a discrete random variable with cumulative distribution function given by:

$$F_X(x) = \begin{cases} 0 & (x < -1) \\ 0.2 & (-1 \leq x < 0) \\ 0.7 & (0 \leq x < 1) \\ 1 & (x \geq 1) \end{cases}$$

   a. Determine the probability function of the random variable X.
   b. Calculate $P(X \geq 1)$.
   c. Evaluate $P(X < 0.5 | X \geq 0)$.

7. The number of cars ordered monthly in a car stand is a random variable X with probability function given by:

x	0	1	2	3	4
$f_X(x)$	0.3	0.3	0.2	0.1	0.1

   a. Determine the cumulative distribution function of random variable X.
   b. Find the minimum number of cars the stand should have monthly so that the probability of meeting all the orders is at least 75%.
   c. In a month in which there are only 2 cars in stock in the stand, evaluate the probability that they are all sold and find the probability distribution of the random variable that represents the cars sold in this month.

8. The number of computers sold per day at Dan's Computer Works is defined by the following probability function:

x	0	1	2	3	4	5	6
$f_X(x)$	0.05	0.1	0.2	0.2	0.2	0.15	0.1

   Evaluate the following probabilities:
   a. $P(3 \leq X \leq 6)$.
   b. $P(X > 3)$.
   c. $P(X \leq 4)$.
   d. $P(2 < X \leq 5)$.

9. A box has 5 balls numbered 1 to 5. Two balls are taken out without replacement. Let X be the random variable that represents the highest value observed.
   a. Determine the probability function of random variable X.
   b. How likely the highest value observed exceeds 3?

10. Find k so that the above functions are probability functions of random variable X:
   a. $f_X(x) = kx \quad (x = 1, 2, \cdots, 10)$;
   b. $f_X(x) = k\left(\frac{1}{5}\right)^x \quad (x = 1, 2, \cdots)$.

11. In a lot of 5 electronic components, two of them are defective. A sample of two components is selected without replacement.

a. Find the percentage of samples without defective components.
b. If X represents the number of defective components in the sample, determine the probability function of X.

12. Let X be a random variable with probability density function given by

$$F_X(x) = \begin{cases} x/4 & (0 < x < 2) \\ 1 - x/4 & (2 < x < 4) \end{cases}$$

a. Determine the cumulative distribution function of the random variable X.
b. Evaluate the probability of X be greater than 3.
c. Calculate $P(X < 3|X > 2)$.
d. Find the cumulative distribution function of the random variable $Y = 8 - 2X$.

13. For each of the following functions find the value of k so that they are probability density functions of a random variable X

a. $f_X(x) = \begin{cases} kx & (0 < x < 1) \\ 2 - x & (1 < x < 2) \end{cases}$

b. $f_X(x) = x^3/4 \quad (0 < x < k)$

c. $f_X(x) = 4x^k \quad (0 < x < 1)$

14. Let $F_X(x) = \begin{cases} 0 & (x < 0) \\ x^2 & (0 \le x < 1/2) \\ -(3x^2 - 6x + 2) & (1/2 \le x < 1) \\ 1 & (x \ge 1) \end{cases}$

a. Prove that $F_X(x)$ is the cumulative distribution function of the random variable X.
b. Determine the probability density function of the random variable X.
c. Calculate $P(X < 3/2)$.
d. Find k such that $P(X \le k) = 1/2$.

15. Let X be a random variable with probability density function given by

$$f_X(x) = x^{-3}/2 \quad (x > 1/2)$$

a. Determine the cumulative distribution function of the random variable X.
b. Evaluate $P(X > 4|X > 2)$.

16. Consider a probability density function defined by

$$f_X(x) = \begin{cases} k(x-1)^2 & (1 < x < 2) \\ 0 & (other\ values\ of\ x) \end{cases}$$

a. Find k. Determine the cumulative distribution function of the random variable X.
b. Concedes that the random experiment associated with random variable X is performed three times independently. Determine the probability function of the random variable Y that represents the number of times when X is bigger than 1.5, in the 3 performances of the random experiment.

17. A machine produces pieces whose lengths show irregularities. The pieces go to waste if it is found a deviation of more than 1 cm from the standard length. Let X be the random variable that represents the difference, in centimeters, of the length of a piece sent to waste from the standard. The probability density function of X is:

$$f_X(x) = \begin{cases} \dfrac{6x+12}{5} & (-2 < x < -1) \\ \dfrac{9}{10x^3} & (1 < x < 3) \\ 0 & (other\ values\ of\ x) \end{cases}$$

a. Find the cumulative distribution function of the random variable X and calculate the percentage of pieces from the waste that has a length greater than the standard.
b. If 10 pieces are selected at random from the waste, what is the probability that half of them have a length greater than the standard.

18. The quantity of wine (dozens of litres) a producer sells per day is a random variable X with cumulative distribution function given by:

$$F_X(x) = \begin{cases} 0 & (x < 0) \\ x^2/50 & (0 \le x < 5) \\ \dfrac{20x - x^2}{50} - 1 & (5 \le x < 10) \\ 1 & (x \ge 10) \end{cases}$$

a. Determine the probability density function of random variable X.
b. Of all the days that he sells more than 50 liters, what is the probability of selling less than 90 liters?
c. If the daily net gain is $Y = 2X - 6$, find the cumulative distribution function of random variable Y. Calculate the proportion of days on which there is a loss.

19. The repair time (dozens hours) of a certain type of computer faults is a random variable X with probability density function given by:

$$f_X(x) = \begin{cases} e^{-x} & (x > 0) \\ 0 & (other\ values\ of\ x) \end{cases}$$

a. Find the cumulative distribution function of random variable X.
b. What upper limit for the recovery time can be established in 40% of cases?
c. Company has established a bonus scheme that assigns 20 euros to workers that make a repair in less than 5 hours and 10 euros to workers that make a repair in more than 5 and less than 8 hours. Find the probability function of random variable Y that represents the value of the bonus.

20. The random variable X represents the ratio of the annual income of the husband and wife of couples residing in a particular region and its probability density function is given by:

$$f_X(x) = \begin{cases} k & (0 < x < 1) \\ \frac{6}{5x^2} & (1 < x < 3) \\ 0 & (other\ values\ of\ x) \end{cases}$$

a. Show that $k = 1/5$ and find the cumulative distribution function of random variable X.
b. Calculate $P(X \leq 2|X > 1)$ and explain its meaning.

21. Admit that for each subject with two hour classes, time of absence from a classroom of a student is a random variable with probability density function:

$$f_X(x) = 1 - x/2 \quad (0 < x > 2)$$

a. What is the probability that a random chosen student attends more than 75% of the class?
b. In a sample of ten students randomly selected, what is the probability of just one of them attend more than 75% of the class?

22. Let X be a random variable with probability density function given by:

$$f_X(x) = \begin{cases} x & (0 < x < 1) \\ 1/2 & (1 \leq x < 2) \end{cases}$$

a. Find the cumulative distribution function of random variable X.
b. Find the cumulative distribution function of random variable $Y = 4X - 2$.
c. Find the cumulative distribution function of random variable

$U = \begin{cases} -1 & (X < 0.5) \\ 1 & (X \geq 0.5) \end{cases}$ and classify it.

23. Let X be a random variable with probability density function given by:

$$f_X(x) = kx^2 \ \ (-1 < x < 1)$$

a. Show that $k = 1.5$.
b. Knowing that X is positive and using the cumulative distribution function calculate the probability that X bigger than 0.5.
c. Find the cumulative distribution function of random variable Y variable which concentrates the negative values of X at zero point and is equal to X for the positive ones. Classify random variable Y.
d. If $Z = (x^3 + 1)/2$, show that its probability density function is $g_Z(z) = 1 \quad (0 < z < 1)$

24. Let X be a random variable with probability density function given by:

$$f_X(x) = \begin{cases} x & (0 < x < 1) \\ 1/2 & (1 \le x < 2) \end{cases}$$

a. Find the cumulative distribution function of random variable X.
b. Calculate $P(X \le 1 | X > 0)$.

25. Classify the following questions as true or false, explaining your choice care-fully.

a. If $f_X(x)$ is the probability function of a discrete random variable then whatever $x \in \mathbb{R},\ 0 \le f_X(x) \le 1$.
b. If $f_X(x)$ is the probability function of a continuous random variable then whatever $x \in \mathbb{R},\ 0 \le f_X(x) \le 1$.
c. If $F_X(x)$ is the cumulative distribution function of a continuous random variable then whatever $x \in \mathbb{R},\ 0 \le F_X(x) \le 1$.
d. If X is a continuous random variable then its probability density function is always lower than or equal to 1.
e. If X is a continuous and symmetric about the origin random variable with cumulative distribution function $F_X(x)$, then $F_X(x) = 1 - F_X(-x)$.
f. If the marginal distributions of X and Y are known it is always possible to determine the joint probability or probability density function of (X, Y).

26. In a certain shop which sells computer components, the daily sales of hard drives of brands X and Y has the following joint probability function:

$y \ \backslash \ x$	0	1	2
0	0.12	0.25	0.13
1	0.05	0.30	0,01
2	0.03	0,10	0.01

a. Find the marginal probability functions of X and Y.
b. What is the probability that, one day, brand X is the best selling brand?
c. Determine the proportion of days on which the same number of disks of each brand are sold.
d. Find the probability function of X on days where exactly 1 disk of Brand Y is sold.

27. The joint probability function of bivariate random variable (X, Y) is given by:

$$f_{X,Y}(x, y) = \frac{x + y}{32} \quad (x = 1, 2; y = 1, 2, 3, 4)$$

a. Determine the marginal probability functions of X and Y.
b. Evaluate $P(X > Y)$ and $P(X = 2Y)$.
c. Find the conditional probability functions and show that X and Y are independent random variables.

28. Exercise 4.79 pp.190 (adopted book)

29. Exercise 4.80 pp.190 (adopted book)

30. The final adjustments to obtain the desired color ink are made firstly by a computer and is completed manually by a skilled worker. The number of color adjustments carried out by the computer varies between one and three. The worker examines the work and proceeds or not with a final manual tuning, if he deems necessary. From past experience it is known that:

- In 60% of cases there is no need for manual tuning;
- In 30% of cases the computer carries out a single tuning of color, and 40% of these no longer need to make any adjustment of color;
- In 60% of cases the computer carries out two tunings of color;
- When the computer carries out three tunings of color a manual tuning of color is always needed.

Determine the joint probability function of the discrete bivariate random variable (X, Y), where X represents the number of computer tunings of color and Y represents the number of manual tunings.

31. Two players A and B, toss a coin. If a head comes player A pays 1 euro to player B; If a tail comes player B pays 1 euro to player A; Game ends when one of the players run out of money. Player A has 2 euros and Player B has 1 euro. Let X be the number of tosses until the end of the game.

a. Find the probability function of random variable X.
b. What is the probability that player B wins?

32. Consider a random vector $(X, Y)$with probability density function given by:

$$f_{X,Y}(x, y) = 2 \quad (0 < x < 1; 0 < y < 1/2)$$

a. Verify that it is a probability density function.
b. Find the marginal probability density function of X and Y. Are X and Y independent?
c. Compute $P(X > 1/2,\ Y < 1/4)$.
d. Compute $P(Y > X)$.

33. It is intended to conduct a survey on the accommodation of families in a region. Some information will be collected through postal survey and part by telephone interview. The Local Institute of Statistics assumes that the joint probability density function of the proportion of families who answered the postal survey (X) and the proportion of families who answered the telephone interview (Y) is given by:

$$f_{X,Y}(x,y) = \frac{3x+5y}{4} \quad (0 < x < 1; 0 < y < 1)$$

a. Find if X and Y independent.
b. Compute the probability that the proportion of families who answered the telephone interview at least doubles the proportion of families who answered the postal survey.

34. Consider the random vector (X, Y), where X represents the length of stay of a student in class and Y the length of time that he is attentive to the subjects taught. The joint probability density function is defined by:

$$f_{X,Y}(x,y) = k \ \ (0 < x < 1; 0 < y < 0.8x)$$

a. Show that $k = 2.5$.
b. What is the probability that the student is attentive to the subjects taught for more than 50% of the length of his stay in class?

35. Let

$$f_{X,Y}(x,y) = x + y \ (0 < x < 1; 0 < y < 1)$$

a. Calculate the $P(X > 1/2, \ \ Y > 1/2)$.
b. Determine the marginal probability density function of X and Y. Are X and Y independent?
c. Find the conditional probability density functions.

36. A firm dedicates to the trade of various items whose sales have random behavior. The monthly sales of items A and B, in monetary units, is a random vector $(X, Y)$with joint probability density function given by:

$$f_{X,Y}(x,y) = k \ \ (0 < x < 2; 0 < y < x)$$

a. Find the value of k.
b. Determine the percentage of months in which both the sales of item A and B are greater than 1 monetary union.
c. Determine the percentage of months in which the sales of item A are greater than 1 monetary union.

37. A company has two factories, A and B, that produce the same good. Consider a random vector (X, Y) that represents the weekly production (in tons) from factories A and B, respectively. It is known that:

$$f_y(y) = \frac{2(5y - y^2)}{33} \quad (1 < y < 4);$$

$$f_{X|Y=y}(x) = \frac{1}{5 - y} \quad (1 < x < 6 - y;\ 1 < y < 4; fixed\ y\ )$$

a. Determine the marginal probability density function of r.v. Y.
b. In a week in which B factory produced 2 tons compute the probability that A factory produces more than 3 tons.
c. Determine, justifying, the joint probability density function of the random vector (X, Y).
d. Calculate the percentage of weeks in which A factory production exceeds B factory production.

38. A person plans to commute in a train that leaves the station between 7:20 and 7:30. Let random variable X represent period of time (in minutes) elapsing between 7:20 and the departure time. X has a probability density function given by:

$$f_X(x) = \frac{10-x}{50} \quad (0 < x < 10)$$

A person arrives at the train station between 7:20 and 7:30. Let random variable Y represent period of time (in minutes) elapsing between 7:20 and the arrival time to the station of that person. Y has a probability density function given by:

$$f_Y(y) = \frac{1}{10} \quad (0 < y < 10)$$

Assume independence between the two random variables.

a. Determine the joint probability density function of the random vector (X, Y).
b. Compute the percentage of days in which the person travels in that train.
c. Calculate the probability of that person having to wait more than 2 minutes until this train departure.

39. John and Jane agreed to meet, between 15:00 and 16:00, to study. Let X be John arrival time and Y Jane arrival time. Random variables X and Y are independent with probability density functions given by:

$$f_X(x) = 1 \quad (15 < x < 16); \quad f_Y(y) = 1 \quad (15 < y < 16)$$

a. Find the joint probability density function.
b. What is the probability that both arrive between 15:30-16:00?
c. Compute the probability that John arrives first.
d. If the first to arrive waits only 15 minutes for the other, calculate the probability that they study together that day.