EXERCISES FOR CHAPTER THREE

1. Let X be a discrete random variable with probability function:

x:	0	1	2	3	4
$f_X(x)$	0.2	0.2	0.1	0.3	0.2

a) Compute $E(X)$ and $Var(X)$.
b) Let $Y = \varphi(X) = 1 - 3X$ and compute $E(Y)$ and $Var(Y)$.
c) If $Z = \varphi(X) = |X - 2|$ calculate $E(Z)$ and $Var(Z)$.

2. A discrete random variable X has a probability function given by:

$$f_X(X) = \frac{x}{10} \qquad (x = 1, 2, 3, 4)$$

a)Compute $E(X)$ and $Var(X)$.

b) Let $Y = \varphi(X) = X^2$ and compute $E(Y)$ and $Var(Y)$.

3. The 1st and 2nd moment about the origin of random variable X are, respectively, equal to 6 and 62. If $Y = \frac{X}{2} + 3$, compute the mean, variance and standard deviation of X.

4. Let X be a continuous random variable with probability density function given by:

$$f_X(x) = \begin{cases} x & (0 < x < 1) \\ 1/2 & (1 < x < 2) \end{cases}$$

a) Compute $E(X)$ and $Var(X)$.
b) Let $Y = \varphi(X) = 1 - 3X$ and compute $E(Y)$ and $Var(Y)$.
c) Determine the 1st and 3rd quartiles.
d) Using the properties of the expected value and of the variance, compute the mean and variance of $Y = 4X - 2$.
e) Compute the mean of the following functions of X:

$$Z = \frac{1}{X} \qquad\qquad U = \begin{cases} -1 & (X < 0.5) \\ 1 & (X \geq 0.5) \end{cases}$$

5. Let X be a continuous random variable with probability density function given by:

$$f_X(x) = \frac{x^2}{42} \qquad (-1 < x < 5)$$

a) Compute the coefficient of variation of the random variable X.
b) Determine the median and the interquartile range.
c) Using the properties of the expected value and of the variance, compute the mean and variance of the random variable $Y = 5 - 3X$.

6. Let X be a continuous random variable with probability density function given by:

$$f_X(x) = \begin{cases} x+2 & (-2 < x < -1) \\ -x & (-1 < x < 0) \end{cases}$$

a) Compute $E(X)$, Median(X) and $Var(X)$.
b) Is the distribution of random variable X symmetric?

7. Let X be a discrete random variable with probability function given by:

$$f_X(x) = \frac{1}{4}\left(\frac{4}{5}\right)^x \quad (x = 1, 2, 3, \cdots)$$

Use the moment generating function to determine the mean and variance of X.

8. Consider $M_X(s)$ the moment generating function of X. Suppose that $R(s) = ln[M_X(s)]$. Show that:

a) $\mu = R'(0)$ b) $\sigma^2 = R''(0)$

9. Consider $M_X(s)$ the moment generating function of X. Show that the m.g.f. of $Y = a + bX \quad (a, b\ constants)$is given by:

$$M_Y(s) = e^{as} M_X(bs)$$

10. Consider a function defined as $f_X(x) = \frac{1}{2}e^{-|x|} \quad (-\infty < x < +\infty)$.

a) Prove that it is a probability density function.
b) Use the moment generating function to determine the mean and variance of X.

11. Let $M_X(s)$ be the m.g.f. of a random variable X. Find the m.g.f. of the random variable $Y = X - \mu$. Show that $M'_X(0) = 0$.

12. In a certain shop which sells computer components, the daily sales of hard drives of brands X and Y has the following joint probability function:

$y \setminus x$	0	1	2
0	0.12	0.25	0.13
1	0.05	0.30	0,01
2	0.03	0,10	0.01

a) Compute the means and variances of X and Y.
b) Analyze the independence of the two random variables and compute the correlation coefficient.
c) Find that $E(Y|X = x)$ is not equal to $E(Y)$. Comment.
d) Compute the mean and variance of $Z = X - Y$.

13. Let (X,Y) be a discrete random vector with joint probability function given by:

$$f_{X,Y}(x,y) = \frac{x+y}{32} \qquad (x = 1,2; y = 1,2,3,4)$$

a) Compute the means and variances of X and Y.
b) Using $E(XY)$, analyze the independence of the two random variables and compute the correlation coefficient.
c) Compute $E(X|Y = y)$.

14. A shopkeeper sells calculators. X, Y are respectively the monthly number of calculators received and the the monthly number of calculators sold. Let (X,Y) be a discrete random vector with joint probability function given by:

$$f_{X,Y}(x,y) = \frac{x+y}{32} \qquad (x = 1,2; y = 1,2,3,4)$$

a) Compute the means and variances of X and Y.
b) Using $E(XY)$, analyze the independence of the two random variables and compute the correlation coefficient.
c) Compute $E(Y|X = 2)$. Interpret the result.
d) Determine the mean and variance of the monthly number of calculators that are not sold.

15. Let (X,Y) be a discrete random vector with joint probability function:

y x	-1	0	1
-1	b	0	c
0	0	a	0
1	c	0	b

a) Find a, b and c such that X *and* Y are not correlated; there is a perfect correlation between them.
b) Compute the mean and variance of $Z = |X - Y|$.

16. Let X and Y be independent random variables with variances σ_X^2 and σ_Y^2. If $Z = X + Y$ and $W = X - Y$ show that $\rho_{ZW} = \frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}$.

17. A company engaged in the trade of various items, whose sales have random behavior. The monthly sales of items A and B, expressed in monetary units, constitute a random vector (X, Y) with joint probability density function given by:

$$f_{X,Y}(x, y = 1/2) \quad (0 < x < 2,\ 0 < y < x)$$

a) Compute the means and variances of X and of Y.
b) Analyze the independence of the two random variables and compute the correlation coefficient.
c) Find the $E(Y|X = 1)$.
d) Compute the mean and variance of total sales of the two articles.

18. Consider the random vector (X, Y), where X represents the length of stay of a student in class and Y the length of time that he is attentive to the subjects taught. The joint probability density function is defined by:

$$f_{X,Y}(x, y) = 2.5 \ (0 < x < 1; 0 < y < 0.8x)$$

Compute the $E(Y|X = x)$ and give an interpretation of the result.

19. Consider the random vector (X, Y) with joint probability density function defined by:

$$f_{X,Y}(x, y) = 8xy \ (0 < x < 1; 0 < y < x)$$

a) Compute the means and variances of X and of Y.
b) Determine the expected value of the product of the two variables and analyze the independence of the two random variables.
c) Compute the correlation coefficient between X and Y.
d) Find the $E(X|Y = y)$.

20. The weekly quantity of feedstock received by a factory is represented by a random variable X and the weekly quantity of feedstock consumed in the production of the same factory is represented by a random variable Y. It is known that:

$f_{Y|X=x}(y) = \frac{3y^2}{x^3} \quad (0 < y < x)$ with a fixed $x \quad (0 < x < 1)$

$f_X(x) = 5x^4 \quad (0 < x < 1).$

a) Compute the mean and standard deviation of the weekly quantity of feedstock received.
b) Calculate $E(Y|X = x)$ and graph it. Determine and interpret the $E(Y|X = 0.75)$.
c) Find the mean and variance of the weekly quantity of feedstock that is not consumed in the factory.
d) Compute the correlation coefficient between X and Y and comment the result.

21. Let (X, Y) be a two-dimensional continuous random variable which represents the weekly sales of products A and B, respectively, and joint probability density function given by:

$$f_{X,Y}(x, y) = e^{-(x+y)} \ (x > 0;\ y > 0)$$

a) Does product B sells more, in average?
b) Find the probability density function of product A sales conditioned by the sales of product B. What can you conclude about the independence of the two random variables?

22. Consider two non-correlated random variables X and Y such that $\sigma_X^2 = \sigma_Y^2$. Let $U = X - Y$ and $V = 2Y$, show that $\rho_{U,V} = -\frac{1}{\sqrt{2}}$.

23. Let (X, Y) be a continuous random vector with joint probability density function given by:

$f_{X,Y}(x, y) = 1 \qquad (0 < x < 1; -x < y < x)$

Show that, in spite of the correlation being nul, the variables are not independent.