1.a) Seja X_n: nº máquinas em bom estado no final do dia "n".

$X_n = 0,1,2,3$, $\quad S=\{0,1,2,3\}$ $\quad \{X_n, n \in \mathbb{N}_0\}$

Seja ξ_n: o nº de máquinas em 3 que avariam no período n, então

$\xi_{n+1} \frown Bin(X_n; 1/2)$, no início de n+1 conhecemos X_n

$$X_{n+1} = \begin{cases} X_n - \xi_{n+1} + 1 & \text{se } 1 \leq X_n \leq 2 \\ 1 & \text{se } X_n = 0 \\ X_n - \xi_{n+1} & \text{se } X_n = 3 \end{cases}$$

Podemos considerar $X_0 = 3$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1/2 & 1/2 & 0 \\ 2 & 0 & 1/4 & 1/2 & 1/4 \\ 3 & 1/8 & 3/8 & 3/8 & 1/8 \end{array}$$

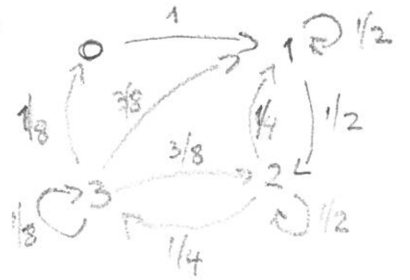
$X_n = 0 \Rightarrow X_{n+1} = 1$ c.p. 1

se $X_n = 1 \Rightarrow X_{n+1} = 2 - \xi_{n+1}$

se $X_n = 2 \Rightarrow X_{n+1} = 3 - \xi_{n+1}$

se $X_n = 3 \Rightarrow X_{n+1} = 3 - \xi_{n+1}$

b)



- todos os estados são comunicantes, classe única.
- Período $d(1)=1 = d(2) = d(3) = d(0)$.
  Cadeia aperiódica
- Estados recorrentes

c) $P_r(X_1 = 3, X_2 = 3) = P_{33}P_{33} = \frac{1}{8}\left(\frac{1}{8}\right) = \frac{1}{8^2} \simeq 0.015625$

d) $P(X_2 = 1) = \sum_{k=0}^{3} P(X_2=1 \mid X_0 = k)\, P(X_0 = k) = \sum_{k=0}^{3} P_{k1}^{(2)} p_k = P_{31}^{(2)} p_3$ se $X_0 = 3$ e $p_3 = 1$

$= P_{31}^{(2)} = \sum_{k=0}^{3} P_{3k}P_{k1} = \frac{1}{8}(1) + \frac{3}{8}\times\frac{1}{2} + \frac{3}{8}\,\frac{1}{4} + \frac{1}{8}\left(\frac{3}{8}\right) = \frac{8+12+6+3}{64} = \frac{29}{64}$

$\simeq 0.453125$

e) Existe distribuição limite

$$\begin{cases} \pi_0 = \frac{1}{8}\pi_3 \\ \pi_2 = 1/2\,\pi_1 + 1/2\,\pi_2 + 3/8\,\pi_3 \\ \pi_3 = \frac{1}{4}\pi_2 + 1/8\,\pi_3 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \end{cases}$$

~~$\pi_1 = \pi_0 + \frac{1}{2}\pi_1 + \frac{1}{4}\pi_2 + \frac{3}{8}\pi_3$~~

$$\begin{cases} \pi_0 = 1/59 \\ \pi_1 = 22/59 \\ \pi_2 = 28/59 \\ \pi_3 = 8/59 \end{cases}$$

R: $\pi_0 + \pi_1 + \pi_2 = 1 - \pi_3 = 1 - 8/59 = 51/59 \simeq 0.8644$

$$\mathbb{Q} = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} -4 & 4 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix} \end{array}$$

$$\begin{tikzcd} \textcircled{0} \arrow[r, shift left, "4"] & \textcircled{1} \arrow[l, shift left, "1"] \arrow[r, shift left, "2"] & \textcircled{2} \arrow[l, shift left, "2"] \end{tikzcd}$$

c) $P_{ij}(h+t) = \sum_{k=0}^{2} P_{ik}(h) P_{kj}(t) = P_{i0}(h) P_{0j}(t) + P_{i1}(h) P_{1j}(t) + P_{i2}(h) P_{2j}(t)$

$= \mu_{i0} P_{0j}(t) + \mu_{i1} P(t) + \mu_{i2} P_{2j}(t) + o(h)$

$P'_{0j}(t) = -\lambda_0 P_{0j}(t) + \lambda_0 P_{1j}(t)$

$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t) + \mu_i P_{i-1,j}(t)$, $i \geq 1$

and replace λ_i, μ_i

d) $\dfrac{\lambda_1}{\lambda_1 + \mu_1} = \dfrac{2}{2+1} = \dfrac{2}{3}$

e) $\begin{cases} -4\pi_0 + \pi_1 = 0 \\ \cancel{4\pi_0 - 3\pi_1 + 2\pi_2 = 0} \\ 2\pi_1 - 2\pi_2 = 0 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$ $(\pi_0, \pi_1, \pi_2) = (1/9, 4/9, 4/9)$

4.(a) Modelo $M|M|2$, $\lambda = 80/60 = 4/3$, $\mu = 1.2^{-1} = 5/6$, $\lambda/\mu = 8/5$, $\rho = \frac{\lambda}{2\mu} = 4/5$

$\theta_k = \frac{1}{k!}\left(\frac{8}{5}\right)^k$, $k = 1, 2$; $\theta_0 = 1$

$\pi_k = \theta_k \pi_0$; $\pi_0 = \left(1 + \frac{8}{5} + \frac{1}{2}\left(\frac{8}{5}\right)^2 \frac{1}{1-4/5}\right)^{-1} = \left(1 + \frac{8}{5} + \frac{8^2}{10}\right)^{-1} = \frac{1}{9} = 1,(1)$

$L_q = \pi_2 \frac{\rho}{(1-\rho)^2} = \frac{1}{2}\left(\frac{8}{5}\right)^2 \frac{1}{9} \times \frac{4}{5} \times \frac{1}{5} = \frac{8^2 \times 2}{45} = 2.84(4)$

b) $W = W_q + \frac{1}{\mu} = \frac{L_q}{\lambda} + \frac{1}{\mu} = \frac{8^2 \times 2}{5 \times 9}\left(\frac{3}{4}\right) + \frac{6}{5} = 3\frac{1}{3}$ //

5. $X_n = X_0 \prod_{i=1}^{n} \xi_i$, com $E[\xi_i] < \infty$ e ξ_i's i.i.d., $\xi_i = e^{\eta_i}$, $\eta_i \sim N(\mu, \sigma^2)$

1. $E[X_n] = x_0 \prod_{i=1}^{n} E[\xi_i] < \infty$

2. $E[X_{n+1} | \mathcal{H}_n] = E[X_n \xi_{n+1} | \mathcal{H}_n] = X_n E[\xi_{n+1}] = X_n E[e^{\eta_i}] = X_n M_\eta(1)$

$= X_n e^{\mu + 1/2 \sigma^2} \implies E[X_{n+1} | \mathcal{H}_n] = X_n$ se $\mu + \frac{1}{2}\sigma^2 = 0$

6. $Cov(W(t), W(s)) = E[W(t) W(s)] - \mu^2 t s$; $E[W(t)] = \mu t$ se $B(0) = 0$.

$E[W(t) W(s)] = \mu^2 s t + \mu \sigma t \, \cancel{E[B(t)]} + \mu \sigma s \, \cancel{E[B(t)]} + \sigma^2 \underbrace{E[B(t) B(s)]}_{s}$, se $B(0) = 0$, $V[B(t)] = 1$

$\implies Cov[W(t), W(s)] = \sigma^2 s$ //

2\. a) $I_j \sim B(1;\theta),\ \forall j$, $M_I(s) = e^{0\times s}(1-\theta) + e^{s}\theta = 1-\theta+\theta e^{s}$

$M_N(s) = e^{\lambda(e^s-1)}$

$\{X(t)\}$: Processo de Poisson composto

$$M_{X(t)}(s) = E[e^{sX(t)}] = E[e^{s\sum_{j=1}^{N(t)} I_j}] = E[E[e^{s\sum^{N(t)} I_j} / N(t)]]$$

$$E[e^{s\sum_1^n I_j} / N(t)=n] = \underbrace{\prod_{i=1}^{n}}_{I_j \text{ indep.}} \underbrace{E[e^{sI_j}] = M_I(s)^n}_{I_j \text{ i.d.}}$$

$$\Rightarrow M_{X(t)}(s) = E[M_I(s)^{N(t)}] = E[e^{N(t)\ln M_I(s)}] =$$

$$= M_{N(t)}(\ln M_I(s)) = e^{\lambda(M_I(s)-1)} = e^{\lambda(\cancel{1}-\theta+\theta e^s-\cancel{1})}$$

$$= e^{\lambda\theta(e^s-1)}$$

b) $\Rightarrow X(t) \sim \text{Poisson}(\lambda\theta)$

c) $T \sim U(0;1)$

$E[N(T)] = E[E[N(T)|T]] = E[\lambda T] = \lambda E[T] = \lambda/2$

$V[N(T)] = E[V(N(T)|T)] + V[E[N(T)|T]]$

$= E[\lambda T] + V[\lambda T] = \lambda\left(\frac{1}{2}\right) + \lambda^2 \frac{1}{12} = \frac{\lambda}{2}\left(1 + \lambda/6\right)$

3\. Seja $X(t)$: nº cabos em funcionamento em t, $S = \{0,1,2\}$

a) $P_{ij}(t,t+s) = P_{ij}(0,t) = P_{ij}(t)$, $i,j = 0,1,2$.

No máximo estão 2 cabos a funcionar, no mínimo zero. Temos um processo de nascimento e morte homogéneo, os tempos de funcionamento e de reparação apenas dependem da amplitude..

b) Tempo de reparação de cada cabo tem distr. Exp(2)

Tempo de espera até avaria de cada cabo tem distr. Exp(1).

- Se estão 2 cabos avariados, "0" a funcionar, o processo muda de estado, logo que o 1º cabo começe a funcionar: $\min\{X_1, X_2\}$, $X_i \sim Exp(2)$ $\Rightarrow \min\{X_1, X_2\} \sim Exp(4)$, $\lambda_0 = 2\times 2 = 4$, $\lambda_1 = 2$, $\lambda_2 = 0$
- Se estão "2" cabos a funcionar, "0" avariados, o processo muda de estado logo que o primeiro avarie: $\min\{Y_1, Y_2\}$, $Y_i \sim Exp(1)$, $\min\{X_1, X_2\} \sim Exp(2)$

$\mu_2 = 2\times 1 = 2$, $\mu_1 = 1$, $\mu_0 = 0$