

Models in Finance - Lecture 2

Master in Actuarial Science

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Martingales

- Idea: a martingale is a stochastic process for which its “current value” is the “optimal estimator” of its expected “future value”. Or:
- Given the stochastic process $\{M_j, j \in \mathbb{N}\}$ and the information \mathcal{F}_n at instant n , then M_n is the best estimator for M_{n+1} .
- A martingale has “no drift” and its expected value remains constant in time.
- Martingale theory is fundamental in modern financial theory: the modern theory of pricing and hedging of financial derivatives is based on martingale theory.

Conditional expectation

- Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{B} \subset \mathcal{F}$ be a σ -algebra.

Definition

The conditional expectation of the integrable r.v. X given \mathcal{B} (or $E(X|\mathcal{B})$) is an integral random variable Z such that

- ① Z is \mathcal{B} -measurable
- ② For each $A \in \mathcal{B}$, we have

$$E(Z\mathbf{1}_A) = E(X\mathbf{1}_A) \quad (1)$$

- If $E[|X|] < \infty$ then $Z = E(X|\mathcal{B})$ exists and is unique

Conditional expectation

- Properties:

1.

$$E(aX + bY|\mathcal{B}) = aE(X|\mathcal{B}) + bE(Y|\mathcal{B}). \quad (2)$$

2.

$$E(E(X|\mathcal{B})) = E(X). \quad (3)$$

3. If X and the σ -algebra \mathcal{B} are independent then:

$$E(X|\mathcal{B}) = E(X) \quad (4)$$

4. If X is \mathcal{B} -measurable (or if $\sigma(X) \subset \mathcal{B}$) then:

$$E(X|\mathcal{B}) = X. \quad (5)$$

5. If Y is \mathcal{B} -measurable (or if $\sigma(Y) \subset \mathcal{B}$) then

$$E(YX|\mathcal{B}) = YE(X|\mathcal{B}) \quad (6)$$

6. Given two σ -algebras $\mathcal{C} \subset \mathcal{B}$ then

$$E(E(X|\mathcal{B})|\mathcal{C}) = E(E(X|\mathcal{C})|\mathcal{B}) = E(X|\mathcal{C}) \quad (7)$$

Conditional expectation

- Given several r.v. Y_1, Y_2, \dots, Y_n , we can consider the conditional expectation

$$E[X|Y_1, Y_2, \dots, Y_n] = E[X|\beta],$$

where β is the σ -algebra generated by Y_1, Y_2, \dots, Y_n .

Martingales

- Let (Ω, \mathcal{F}, P) be a probability space and $\{\mathcal{F}_n, n \geq 0\}$ be a sequence of σ -algebras such that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_n \subset \dots \subset \mathcal{F} \quad (8)$$

The sequence $\{\mathcal{F}_n, n \geq 0\}$ is called a filtration

- Filtration \approx “information flow”.

Definition

$M = \{M_n; n \geq 0\}$ (in discrete time) is a martingale with respect to filtration $\{\mathcal{F}_n, n \geq 0\}$ if:

- For each n , M_n is a \mathcal{F}_n -measurable r.v. (i.e., M is a stochastic process adapted to the filtration $\{\mathcal{F}_n, n \geq 0\}$).
- For each n , $E[|M_n|] < \infty$.
- For each n , we have:

$$E[M_{n+1} | \mathcal{F}_n] = M_n. \quad (9)$$

Martingales

- If we consider the filtration $\mathcal{F}_n = \sigma(M_0, M_1, \dots, M_n)$, then we say that $M = \{M_n; n \geq 0\}$ is a martingale (with respect to this filtration) if

- For each n , $E[|M_n|] < \infty$.
- For each n , we have:

$$E[M_{n+1} | \mathcal{F}_n] = M_n. \quad (10)$$

- Properties: It is easy to show that if $M = \{M_n; n \geq 0\}$ is a martingale then

- $E[M_n] = E[M_0]$ for all $n \geq 1$.
- $E[M_n | \mathcal{F}_k] = M_k$ for all $n \geq k$.

- Exercise: Prove properties 1. and 2. above.

Martingales

- idea: the “current value” M_k of a martingale is the “optimal estimator” of its “future value” M_n .
- martingale and risk neutral probability measure: If the discounted price of a financial asset is a martingale when calculated using a particular probability distribution, then this probability distribution is called a “risk-neutral” probability measure (meaning that the price has no “drift”).

- Example: Assume that share S has a price process S_n and a discounted price process

$$\tilde{S}_n = e^{-rn} S_n, \quad (11)$$

where r is the risk-free interest rate. If we assume that for a probability measure Q , the process \tilde{S}_n is a martingale, then under Q , we have that

$$E_Q \left[\tilde{S}_{n+1} | \tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_n \right] = \tilde{S}_n.$$

Since \tilde{S}_n is known (it is measurable) with respect to $\sigma(\tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_n)$, then by property (5), we have:

$$\begin{aligned} E_Q \left[\frac{e^{-r(n+1)} S_{n+1}}{e^{-rn} S_n} | \tilde{S}_0, \tilde{S}_1, \dots, \tilde{S}_n \right] &= 1 \\ \iff E_Q \left[\frac{S_{n+1}}{S_n} | S_0, S_1, \dots, S_n \right] &= e^r. \end{aligned}$$

Therefore, the expected return in period from time n to time $n + 1$ is the risk-free rate: that is why Q is called a risk-neutral measure.

Martingales in continuous time

- Probability space (Ω, \mathcal{F}, P) and family of σ -algebras $\{\mathcal{F}_t, t \geq 0\}$ such that

$$\mathcal{F}_s \subset \mathcal{F}_t, \quad 0 \leq s \leq t. \quad (12)$$

The family $\{\mathcal{F}_t, t \geq 0\}$ is called a filtration

- Let \mathcal{F}_t^X be the σ -algebra generated by process X on the interval $[0, t]$, i.e. $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Then \mathcal{F}_t^X is the “information generated by X on interval $[0, t]$ ” or “history of the process X up until time t ”.
- $A \in \mathcal{F}_t^X$ means that it is possible to decide if event A occurred or not, based on the observation of the paths of the process X on $[0, t]$.
- Example: If $A = \{\omega : X(5) > 1\}$ then $A \in \mathcal{F}_5^X$ but $A \notin \mathcal{F}_4^X$.
- A stochastic process Y is said to be adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if Y_t is \mathcal{F}_t -measurable for all t .
- If $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$ is the filtration generated by X , then any continuous function of X_t is adapted to \mathcal{F}_t^X .

Martingales in continuous time

- Key properties:
 - ① $E\{E[X|\mathcal{F}_t]\} = E[X]$.
 - ② If X is \mathcal{F}_t -measurable then $E[X|\mathcal{F}_t] = X$.
 - ③ If Y is \mathcal{F}_t -measurable and bounded then $E[XY|\mathcal{F}_t] = YE[X|\mathcal{F}_t]$.
 - ④ If X is independent of \mathcal{F}_t then $E[X|\mathcal{F}_t] = E[X]$.

Martingales in continuous time

Definition

A stochastic process $M = \{M_t; t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_t, t \geq 0\}$ if:

- ① For each $t \geq 0$, M_t is a \mathcal{F}_t -measurable r.v. (i.e., M is adapted to $\{\mathcal{F}_t, t \geq 0\}$).
- ② For each $t \geq 0$, $E[|M_t|] < \infty$.
- ③ For each $s \leq t$,

$$E[M_t | \mathcal{F}_s] = M_s. \quad (13)$$

Martingales in continuous time

- cond. (3) $\iff E[M_t - M_s | \mathcal{F}_s] = 0$.
- If $t \in [0, T]$ then $M_t = E[M_T | \mathcal{F}_t]$.
- cond. (3) $\implies E[M_t] = E[M_0]$ for all t .

Martingales in continuous time

- Consider a Bm $B = \{B_t; t \geq 0\}$ defined on (Ω, \mathcal{F}, P) and

$$\mathcal{F}_t^B = \sigma \{B_s, s \leq t\}. \quad (14)$$

Proposition: The following processes are \mathcal{F}_t^B -martingales:

- ① B_t .
- ② $B_t^2 - t$.
- ③ $\exp\left(aB_t - \frac{a^2 t}{2}\right)$. (Exercise: prove that this process is a martingale).

Martingales in continuous time

Proof.

1. B_t is \mathcal{F}_t^B -measurable and therefore it is adapted. $E[|B_t|] < \infty$ (why?) Moreover $B_t - B_s$ is independent of \mathcal{F}_s^B (why?). Hence (why?)

$$E[B_t - B_s | \mathcal{F}_s^B] = E[B_t - B_s] = 0.$$

2. Clearly, $B_t^2 - t$ is \mathcal{F}_t^B -measurable and adapted (why?) and $E[|B_t^2 - t|] < \infty$. By the properties of the conditional expectation

$$\begin{aligned} E[B_t^2 - t | \mathcal{F}_s^B] &= E[(B_t - B_s + B_s)^2 | \mathcal{F}_s^B] - t \\ &= E[(B_t - B_s)^2] + 2B_s E[B_t - B_s | \mathcal{F}_s^B] + B_s^2 - t \\ &= t - s + B_s^2 - t = B_s^2 - s. \end{aligned}$$

□

Martingales in continuous time

- Exercise: Prove that $\exp\left(aB_t - \frac{a^2 t}{2}\right)$ is a $\{\mathcal{F}_t^B, t \geq 0\}$ -martingale.