Mathematical Finance

Retake exam 2010

All notation must be clear. Arguments should be complete.

1. Consider a model with two risky assets S^1 and S^2. The dynamics are given by

$$\begin{aligned} dS_t^1 &= \alpha_1 S_t^1 dt + \sigma_1 S_t^1 dW_t, \\ dS_t^2 &= \alpha_2 S_t^2 dt + \sigma_2 S_t^2 dW_t, \\ dB_t &= rB_t dt. \end{aligned}$$

   where W is a scalar Wiener process (the same W is driving both assets). Use the first fundamental theorem and Girsanov to derive necessary and sufficient conditions for absence of arbitrage. Discuss these conditions in economic terms.

2. Consider a standard Black-Scholes model for a stock where

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= rB_t dt \end{aligned}$$

   and W is a P-Wiener process. Now consider a **digital call** with strike K and exercise date T. This contract will give you the fixed amount A if $S_T \leq K$, and zero otherwise. Compute the price at time $t = 0$ for the digital call.

3. Let the stock prices S^1 and S^2 be given as the solutions to the following system of SDE:s.

$$\begin{aligned} dS_t^1 &= \alpha S_t^1 dt + \delta S_t^1 dW_t, \quad S_0^1 = s_1, \\ dS_t^2 &= \beta S_t^2 dt + \gamma S_t^2 dV, \quad S_0^2 = s_2, \end{aligned}$$

   The Wiener processes W and V are assumed to be independent. The parameters $\alpha,\ \delta,\ \gamma,\ \beta$ are assumed to be known and constant. Your task is to price a **maximum option**. This T-claim is defined by

$$X = \max\left[S_T^1, S_T^2\right]$$

   The pricing function for a European call option in the Black-Scholes model is assumed to be known, and is denoted by $c(s, t; K, \sigma, r, T)$

where σ is the volatility, K is the strike price and r is the short rate. You are allowed to express your answer in terms of this function, with properly derived values for K, σ and r.

4. Consider a complete financial market with constant short rate r, and a (unique) risk neutral martingale measure Q. Use the martingale approach to derive a formula for the optimal wealth profile which maximizes the expected utility

$$E\left[U(X_T)\right]$$

where X is the value process for the portfolio and U is the utility function.