Mathematics 2

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Syllabus

- **Complements of linear algebra**. Eigenvalues and eigenvectors. Quadratic forms.
- **Functions of several variables**. General concepts, domain, image, geometric representation. Topology in $\mathbb{R}^n$. Continuity. Partial derivatives and differentiability. Constrained and unconstrained optimization. Multiple integrals.
- **Differential equations**.Generalities. Existence and uniqueness results. First order equations (linear, separable). Higher order equations (linear, with constant coefficients).
- **Difference equations**.Generalities. First and second order difference equations with constant coefficients.

Assessment

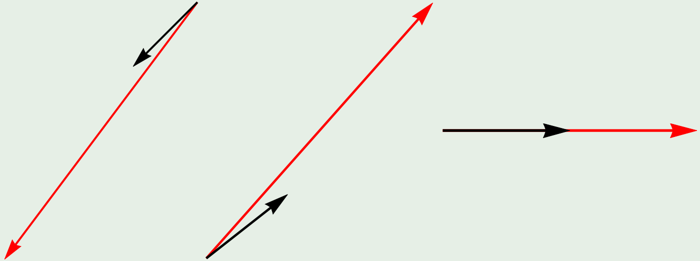
- **Normal period**
  MT: Midterm exam covering the first half of the syllabus.
  F: Final exam, covering the second half of the syllabus.
  ON: Online quizzes along the semester.
  Grade = 0.40 * MT + 0.40 * F + 0.20 * ON
  During the final exam students are given the chance of improving (MT).
- **Repeat period** Students can choose to be evaluated as in the normal exem period or just by a final exam covering the whole syllabus.
- NOTE 1: Attendance to classes is mandatory for students using assessment during the semester. Only students attending at least 75% of both theoretical and exercise classes will be scored at (MT) and (ON).
- NOTE 2: (MT) and (F) have a minimum grade of 8.0/20

Eigenvectors and eigenvalues

If we fix a basis on $\mathbb{R}^n$, a square matrix $A \in \mathbb{R}^{n \times n}$ can be seen as an application from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Example

Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$. If we consider any vector $\boldsymbol{u}$ and compute its image $\boldsymbol{Au}$, this new vector may or may not have the same direction as $\boldsymbol{u}$.



Eigenvectors and eigenvalues

When $\boldsymbol{u}$ and $\boldsymbol{Au}$ have the same direction, we say that $\boldsymbol{u}$ is an **eigenvector** of A. If $\boldsymbol{u}$ and $A\boldsymbol{u}$ have the same direction, there exists λ such that $A\boldsymbol{u} = \lambda \boldsymbol{u}$. The number λ is an amplification or reduction factor called an **eigenvalue** of A.

Definition

Let A be a square matrix of order n. if there exists $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$ such that $Au = \lambda u$ we say that λ is an **eigenvalue** of A and u is an **eigenvector** associated to that eigenvalue.

Proposition

Given an eigenvector of a square matrix A, there is one and only one eigenvalue associated to it.

Eigenvectors and eigenvalues

Proposition

If u is an eigenvector associated to an eigenvalue λ, any multiple of u is also an eigenvector associated to λ.

If $\boldsymbol{u} \neq 0$ is an eigenvalue of A and λ is its eigenvalue, then $A\boldsymbol{u} = \lambda\boldsymbol{u} \quad \Leftrightarrow (A - \lambda I)\boldsymbol{u} = 0$ This homogeneous system can only have nonzero solutions if the system matrix is not invertible.

Proposition

λ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$ if and only if

$$p(\lambda) := \det(A - \lambda I) = 0$$

$p(\lambda)$ is a polynomial of degree n in λ and is called the **characteristic polynomial** *of A.*

Definition (Algebraic multiplicity)

The algebraic multiplicity of an eigenvalue is its multiplicity as a root of $p(\lambda)$.

Example

Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

The eigenvalues of A are the solutions of the equation

$$\det(A - \lambda I) = 0 \Leftrightarrow (1-\lambda)^2 = 0 \Leftrightarrow \lambda = 1 \vee \lambda = 1.$$

The eigenvalue $\lambda = 1$ has multiplicity 2 as a root of the polynomial characteristic e so we say that it has **algebraic multiplicity** 2.

Computing eigenvectors

We compute the eigenvectors associated to an eigenvalue λ by solving the undetermined system $(A - \lambda I)u = 0$.

The degree of indetermination of this system, given by $gm = n - \text{rank}(A - I\lambda)$, corresponds to the maximum number of linearly independent eigenvector that can be associated to λ.

Definition (Geometric multiplicity)

The geometric multiplicity of an eigenvalue λ is given by $gm = n - \text{rank}(A - I\lambda)$.

Remark

If $1 \leq gm \leq n$ the eigenspace of λ has dimension gm. This means that any eigenvector associated to λ can be obtained as a linear combination of gm fixed eigenvectors associated to λ.

Example

Let us determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 4 & -10 & 10 \\ 0 & 6 & -2 \\ 0 & -2 & 6 \end{pmatrix}$$

We start by determining the eigenvalues as the solutions of

$$\det(A - \lambda I) = 0 \Leftrightarrow \quad (4-\lambda)\left[(6-\lambda)^2 - 4\right]$$
$$\lambda = 4 \vee \lambda = 8 \vee \lambda = 4$$

The eigenvalues are $\lambda = 4$ (with alg. multiplicity 2) and $\lambda = 8$ (a. m. $= 1$). The eigenvectors associated to $\lambda = 4$ are the nontrivial solutions of $(A - 4I)u = 0$.

Example (cont.)

$$(A-4I)u=0 \Leftrightarrow \begin{cases} -10u_2+10u_3=0 \\ 2u_2-2u_3=0 \\ -2u_2+2u_3=0 \end{cases} \Leftrightarrow u_2=u_3$$

This means that any vector (u_1, u_2, u_3) such that $u_2 = u_3$ is an eigenvector associated to $\lambda = 4$. The value of u_1 can be choosen arbitralily, say $u_1 = t$, and if the choose $u_2 = s$ then we must also set $u_3 = s$. So u is an eigenvector if

$$u=(t,s,s), \quad t,s\in\mathbb{R} \quad \Leftrightarrow \quad u=t(1,0,0)+s(0,1,1), s^2+t^2\neq 0$$

The geometric multiplicity of $\lambda = 4$ is two. Any eigenvector associated to $\lambda = 4$ can be writen as a linear combination of the vectors $(1, 0, 0)$ and $(0, 1, 1)$.
Running similar calculations we can check that the eigenvectors associated to $\lambda = 8$ are of the form $u = t(5, -1, 1), t \neq 0$.

Properties

Proposition

Let $A \in \mathbb{R}^{n\times n}$.

- *If* A *is an upper or lower triangular matrix, the eigenvalues are the diagonal elements of* A.
- *If* $\lambda_1, \cdots \lambda_n$ *are* n *real eigenvalues of* A *then* $\det(A) = \lambda_1 \times \cdots \times \lambda_n$.
- λ *is an eigenvalue of* A *if and only if* λ *is an eigenvalue of the transposed matrix* A'.
- *If* A *in invertible* λ *is an eigenvalue of* A *if and only if* $1/\lambda$ *is an eigenvalue of* A^{-1}.
- *Any two eigenvectors associated to the same eigenvalue are linearly independent.*
- *A set of* k *eigenvectors associated to* k *distinct eigenvalues is linearly independent.*

Quadratic forms

Definition

A quadratic form in n variables is any function $Q : \mathbb{R}^n \to \mathbb{R}$ expressed as a sum of second order terms in the variables $x_1, \cdots, x_n$.

$$\begin{aligned} Q(\boldsymbol{x}) = \quad & c_{11}x_1x_1 + c_{12}x_1x_2 + \cdots + c_{1n}x_1x_n \\ & + c_{21}x_2x_1 + c_{22}x_2x_2 + \cdots + c_{2n}x_2x_n \\ & \vdots \\ & c_{n1}x_nx_1 + c_{n2}x_nx_2 + \cdots + c_{nn}x_nx_n \\ = \quad & \sum_{i,j=1}^{n} c_{ij}x_ix_j = \sum_{i\leq j}^{n} b_{ij}x_ix_j \end{aligned}$$

where $b_{ij} = c_{ij} + c_{ji}$ if $i \neq j$ and $b_{ii} = c_{ii}$.

Proposition

Any quadratic form in n variables can be writen in the form $Q(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$, where A is a square matrix of order n, where $a_{ii} = b_{ii}$ and $a_{ij} + a_{ji} = b_{ij}$. If we require that $A^T = A$ this representation is unique and we have $a_{ii} = b_{ii}$, $a_{ij} = b_{ij}/2, i < j$ and $a_{ij} = a_{ji}$.

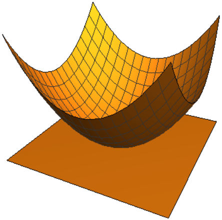
Example

Let $Q(x_1, x_2, x_3) = 2x_1^2 + 3x_1x_3 + 3x_2^2 + 2x_2x_3 + 4x_3^2$. We have

$$Q(x_1, x_2, x_3) = (x_1 x_2 x_3) \begin{pmatrix} 2 & 0 & \frac{3}{2} \\ 0 & 3 & \frac{2}{2} \\ \frac{3}{2} & \frac{2}{2} & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

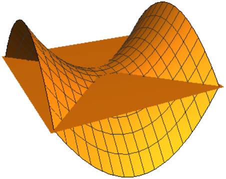
Classification of quadratic forms

It is of great importance in many applications to classify quadratic forms with respect to their sign.



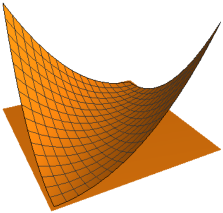
$Q(x_1, x_2) = x_1^2 + x_2^2$

Positive for all $\boldsymbol{x} \neq 0$.



$Q(x_1, x_2) = x_1^2 - x_2^2$

Sign is not fixed.



$Q(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2$

Positive or null.

Definition

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. We say it is:

- **Positive definite** if $Q(\boldsymbol{x}) > 0, \quad \forall x \neq 0$.
- **Negative definite** if $Q(\boldsymbol{x}) < 0, \quad \forall x \neq 0$.
- **Positive semi-definite** if $Q(\boldsymbol{x}) \geq 0, \forall x$ and there is some $\boldsymbol{y} \neq 0$ such that $Q(\boldsymbol{y}) = 0$.
- **Negative semi-definite** if $Q(\boldsymbol{x}) \leq 0, \forall x$ and there is some $\boldsymbol{y} \neq 0$ such that $Q(\boldsymbol{y}) = 0$.
- **Indefinite** if there are $\boldsymbol{x}, \boldsymbol{y}$ such that $Q(\boldsymbol{x}) > 0$ and $Q(\boldsymbol{y}) < 0$.

These definitions extend naturally to symmetric matrices.

Definition

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. We say it is:

- **Positive definite** if $\boldsymbol{x}^t A\boldsymbol{x} > 0, \quad \forall \boldsymbol{x} \neq 0.$
- **Negative definite** if $\boldsymbol{x}^t A\boldsymbol{x} < 0, \quad \forall \boldsymbol{x} \neq 0.$
- **Positive semi-definite** if $\boldsymbol{x}^t A\boldsymbol{x} \geq 0, \forall \boldsymbol{x}$ and there is some $\boldsymbol{y} \neq 0$ such that $\boldsymbol{y}^t A\boldsymbol{y} = 0.$
- **Negative semi-definite** if $\boldsymbol{x}^t A\boldsymbol{x} \leq 0, \forall \boldsymbol{x}$ and there is some $\boldsymbol{y} \neq 0$ such that $\boldsymbol{y}^t A\boldsymbol{y} = 0$
- **Indefinite** if there are $\boldsymbol{x}, \boldsymbol{y}$ such that $\boldsymbol{x}^t A\boldsymbol{x} > 0$ and $\boldsymbol{y}^t A\boldsymbol{y} < 0.$

Classification of Quadratic forms

Theorem

Let $A \in \mathbb{R}^{n\times n}$ *be a symmetric matrix with real eigenvalues* $\lambda_i, \quad i = 1, \cdots n.$ *Then*

- A *is Positive definite iif all eigenvalues are positive.*
- A *is Negative Definite iif all eigenvalues are negative.*
- A *is Positive semi-definite iif there is at least one null eigenvalue, while others are nonnegative.*
- A *is Negative semi-definite iif there is at least one null eigenvalue, while others are nonpositive.*
- A *is Indefinite if there are eigenvalues of different signs.*

Note

The eigenvalues symmetric matrices are always real numbers.

Classification of Quadratic forms

Definition (Principal submatrix)

$B \in \mathbb{R}^{k \times k}(k < n)$ is called a principal submatrix of $A \in \mathbb{R}^{n \times n}$ if it is obtained by removing k rows of A, together with the columns having the same index.

Definition (Primary principal submatrix)

A primary principal submatrix is a principal submatrix obtained by removing the last k rows and columns $(k = 0, \cdots, n-1)$

Definition (Principal minors)

We define the principal minors of a matrix $A \in \mathbb{R}^{n \times n}$ as the determinants of the primary principal minors.

Principal Minors

$$\Delta_1 = \det\left(a_{11}\right), \quad \Delta_2 = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

$$\Delta_3 = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \Delta_4 = \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

$$\cdots, \quad \Delta_n = \det A$$

Classification of Quadratic forms

Theorem (Classification by principal minors)

Let A be a symmetric matrix of order n. Then

- *A is positive definite if and only if* $\Delta_1 > 0, \Delta_2 > 0, \cdots, \Delta_n > 0$.
- *A is negative definite if and only if* $\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \cdots$.
- *If $\Delta_n = \det A \neq 0$ and the principal minors do not verify the previous conditions, the matrix is indefinite.*

If $\det A = 0$ the previous result does not help us classifying the matrix and it can be either semi-definite or indefinite. In that case we must compute the eigenvalues.

Example

Let us classify the quadratic form

$$Q(x, y, z) = x^2 + 2xy + 4xz - 6yz + 3z^2.$$

The associated symmetric matrix is

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -3 & 3 \end{pmatrix}$$

The principal minors are given by

$$\Delta_1 = 1 > 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$$

$$\Delta_3 = |A| = -24 \neq 0$$

We are in the third case mentioned in the previous proposition and the quadratic form is therefore undetermined. In fact we can directly check that $Q(0, 0, 1) = 3 > 0$ and $Q(0, 1, 1) = -3 < 0$.