Functions of several variables

In general, a **function** $f : A \to B$ is a correspondence that assigns to each element in A a unique element in B.

The set A, where the correspondence is defined is called the **Domain** of f, which we denote by D_f.

The **Image** of f is the subset of B defined by

$$Im(D_f) = \{y \in B : y = f(x), x \in A\}.$$

We will study the particular situation where both A and B are real

vectors of given dimensions

Functions of several variables

$$f : A \subset \mathbb{R}^n \to \mathbb{R}^m, \qquad (x_1, x_2, \cdots, x_n) \stackrel{f}{\longmapsto} (y_1, y_2, \cdots, y_n)$$

Example ($f(x,y) = x^2 + y^2$)

This expression can be computed for any x, y, and so the domain is $D_f = \mathbb{R}^2$. Because f can take any nonnegative value, the image of f is $Im(D_f) = \mathbb{R}_0^+$.

Example ($f(x,y) = \sqrt{x^2 + y^2 - 9}$)

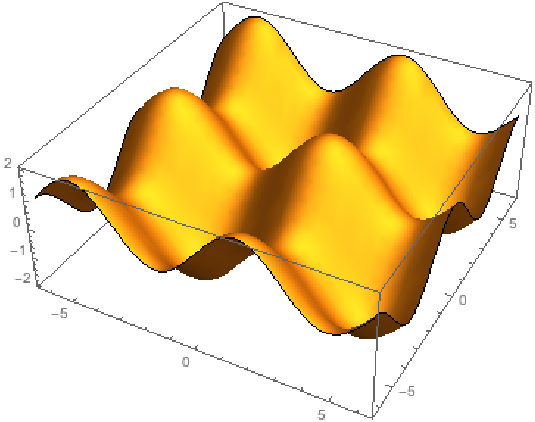
$$D_f = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 9 \geq 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 9\}$$

The values of f can only be computed if (x,y) is in a circle of radius 3 centered in $(0,0)$. $Im(D_f) = \mathbb{R}_0^+$.

Geometric representation

The **Graphic** of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the subset of $\mathbb{R}^{n+1}$ defined by

$$\mathrm{Graph}(f) = \{(\boldsymbol{x}, f(\boldsymbol{x})) \in \mathbb{R} \times \mathbb{R}^n : \boldsymbol{x} \in D_f\}$$



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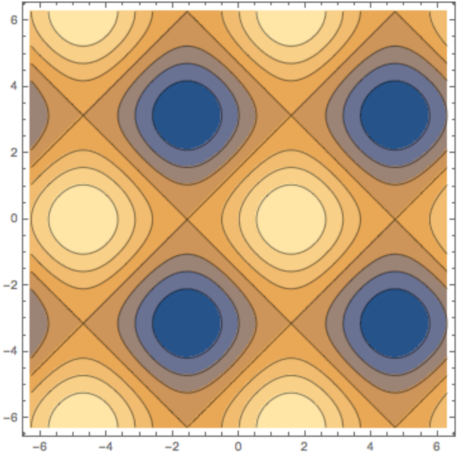
Levelsets

$$L_\alpha(f) = \{x \in D_f : f(\boldsymbol{x}) = \alpha\}$$

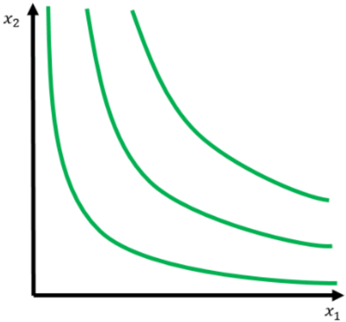


The function f has a constant values on each line on the graphic. The color codes are related to the value of f.

Levelsets in real world applications

Levelsets in real world applications

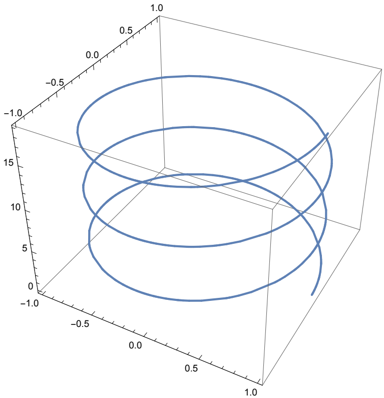
Cobb-Douglas utility function



$$u(x_1, x_2) = x_1^c x_2^d$$
$$c, d > 0$$

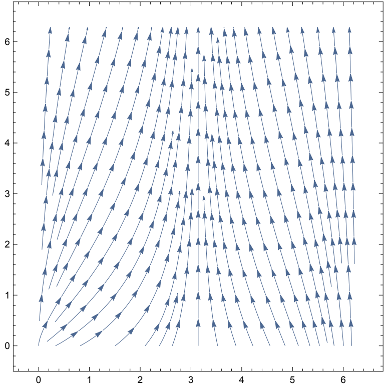
- Gives monotone, convex preferences
- Easy to work with

Other examples



$$f : \mathbb{R} \to \mathbb{R}^3$$

$$t \longmapsto (\cos t, \sin t, t)$$



$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x, y) \longmapsto (2 \sin x, x + y)$$

Topology in $\mathbb{R}^n$

Definition (Distance)

An application $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a **distance** if

- $d(x, y) = 0 \Leftrightarrow x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, z) \leq d(x, y) + d(y, z)$.

We will use the Euclidean distance, given by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

Other definitions of distance

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|$$

$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \cdots, |x_n - y_n|\} = \max_{i=1,\cdots,n} |x_i - y_i|$$

Example $(x = (1,3), \quad y = (-1,5))$

$$d_1(x,y) = |1-(-1)| + |3-5| = 4$$

$$d_\infty(x,y) = \max\{|1-(-1)|, |3-5|\} = \max\{2,2\} = 2$$

$$d_2(x,y) = \sqrt{(1-(-1))^2 + (3-5)^2} = \sqrt{8} = 2\sqrt{2}$$

Definition

We define a ball (or neighborhood) centered in $\boldsymbol{x}$ and with radius $\varepsilon > 0$ as

$$B_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : d(\boldsymbol{x}, \boldsymbol{y}) < \varepsilon\}.$$

$B_\varepsilon(\boldsymbol{x})$ is also called an **open ball**.

Example

The open balls in $\mathbb{R}$ are given by

$$B_\varepsilon(x) = \{y \in \mathbb{R} : |x - y| < \varepsilon\} =]x - \varepsilon, x + \varepsilon[$$

and the open balls in $\mathbb{R}^2$ are given by

$$B_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^2 : (x_1 - y_1)^2 + (x_2 - y_2)^2 < \varepsilon\}$$

Definition

Let $\Omega \subseteq \mathbb{R}^n$ and $\boldsymbol{a} \in \mathbb{R}^n$.

- $\boldsymbol{a}$ is an **interior point** to Ω if there exists some $\varepsilon > 0$ such that $B_\varepsilon(\boldsymbol{a}) \subset \Omega$.
- $\boldsymbol{a}$ is an **exterior point** to Ω if there exists some $\varepsilon > 0$ such that $B_\varepsilon(\boldsymbol{a}) \subset \Omega^C$.
- $\boldsymbol{a}$ is a **boundary point** to Ω if for every $\varepsilon > 0$ the open ball $B_\varepsilon(\boldsymbol{a})$ contains points of both Ω and Ω^C.

Definition (Interior, exterior, boundary)

Let $\Omega \subseteq \mathbb{R}^n$, we define the sets:

- $Int(\Omega) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \text{ is interior to } \Omega\}$
- $Ext(\Omega) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \text{ is exterior to } \Omega\}$
- $Bdy(\Omega) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \text{ is a boundary point to } \Omega\}$

Proposition

Let $\Omega \subseteq \mathbb{R}^n$. Then, every point $\boldsymbol{x} \in \mathbb{R}^n$ belongs exactly to one of the sets $Int(\Omega$, $Ext(\Omega)$ or $Bdy(\Omega)$. We have

$$Int(\Omega) \cup Ext(\Omega) \cup Bdy(\Omega) = \mathbb{R}^n$$

$$Int(\Omega) \cap Ext(\Omega) = \emptyset, Int(\Omega) \cap Bdy(\Omega) = \emptyset, Ext(\Omega) \cap Bdy(\Omega) = \emptyset.$$

Definition (Closure)

We define the **closure** or **adherence** of a set $\Omega \subseteq \mathbb{R}^n$, and denote it by $\overline{\Omega}$, as the set of all interior and boundary points:

$$\overline{\Omega} := Int(\Omega) \cup Bdy(\Omega)$$

Open and closed sets

Definition

Let $\Omega \subseteq \mathbb{R}^n$.

- Ω is an **open set** if it coincides with its interior, $Int(\Omega) = \Omega$.
- Ω is an **closed set** if it coincides with its adherence, $\overline{\Omega} = \Omega$.

Remark

- *Some sets are neither open nor closed.*
- *Some sets are both open and closed.*

Definition (Limit points, limit set)

Let $\Omega \subseteq \mathbb{R}^n$. The **limit set** of Ω, denoted by Ω', is the set of all points $\boldsymbol{x}$ that have elements of Ω in every neighbourhood, i.e.

$$\boldsymbol{x} \in \Omega' \text{ if } \forall \varepsilon > 0 (B_\varepsilon(\boldsymbol{x}) \setminus \boldsymbol{x}) \cap \Omega \neq \emptyset$$

Definition (Isolated point)

Let $\Omega \subseteq \mathbb{R}^n$ and $\boldsymbol{x} \in \Omega$. We say that $\boldsymbol{x}$ is an **isolated point** if for some $\varepsilon > 0$ we have $B_\varepsilon(\boldsymbol{x}) \cap \Omega = \{x\}$.

Proposition

$$\overline{\Omega} = \Omega' \cup \{\boldsymbol{x} : \boldsymbol{x} \text{ is an isolated point}\}$$

Definition (Bounded set)

A set $\Omega \subseteq \mathbb{R}^n$ is **bounded** if it is contained in some open ball, i.e. if there exist $\boldsymbol{x} \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $\Omega \subset B_\varepsilon(\boldsymbol{x})$.

Definition (Compact set)

A set $\Omega \subseteq \mathbb{R}^n$ is **compact** if it is closed and bounded.

Sequences in $\mathbb{R}^m$

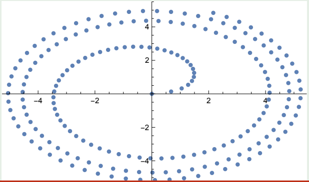
Definition (Sequence)

A sequence in $\mathbb{R}^m$ is an ordered (infinite) list of elements of $\mathbb{R}^m$:

$$
\begin{aligned}
\boldsymbol{u}_1 &= (u_{1,1}, u_{i,2}, \cdots, u_{1,m}) \\
\boldsymbol{u}_2 &= (u_{2,1}, u_{2,2}, \cdots, u_{2,m}) \\
\cdots & \\
\boldsymbol{u}_n &= (u_{n,1}, u_{n,2}, \cdots, u_{n,m}) \\
\cdots &
\end{aligned}
$$

Example

$$\boldsymbol{u}_n = \left(\log n \cos\frac{n}{10}, \log n \sin\frac{n}{10}\right)$$



Convergent sequences

Definition

A sequence $(\boldsymbol{u}_n) \subset \mathbb{R}^m$ is said to be **convergent** to $\boldsymbol{a} \in \mathbb{R}^m$ if

$$\forall \varepsilon > 0 \quad \exists p \in \mathbb{N} : n > p \Rightarrow d(\boldsymbol{u}_n, \boldsymbol{a}) < \varepsilon.$$

In this case we write $\boldsymbol{u}_n \to \boldsymbol{a}$ or $\lim \boldsymbol{u}_n = \boldsymbol{a}$ or $\lim\limits_{n\to\infty} \boldsymbol{u}_n = \boldsymbol{a}$.

Proposition

A sequence $(\boldsymbol{u}_n) \subset \mathbb{R}^m$ converges to $\boldsymbol{a} \in \mathbb{R}^m$ if and only if the (real) sequence $\|\boldsymbol{u}_n - \boldsymbol{a}\|$ converges to zero in $\mathbb{R}$, i.e.

$$\lim \boldsymbol{u}_n = \boldsymbol{a} \Leftrightarrow \|\boldsymbol{u}_n - \boldsymbol{a}\| \to 0.$$

Remark (Norm of a vector $\boldsymbol{x} \in \mathbb{R}^m$)

$$\|\boldsymbol{x}\| = d(\boldsymbol{x}, 0) = \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}.$$

Convergent sequences

The previous result is actually equilalent to the following:

Proposition

Let $(\boldsymbol{u}_n)$ *be a sequence in* $\mathbb{R}^m$ *such that* $\boldsymbol{u}_n = (u_{n,1}, u_{n,2}, \cdots, u_{n,m})$. *Then*

$$\lim \boldsymbol{u}_n = \boldsymbol{a} \Leftrightarrow \lim u_{n,1} = a_1, \quad \lim u_{n,2} = a_2, \cdots, \lim u_{n,m} = a_m.$$

Example

The previou proposition tells us that the limit of a sequence ir $\mathbb{R}^m$ can be computed component by component. For instance,

$$\lim \left(\frac{n^2+1}{2n^2+3}, \left(1+\frac{1}{n}\right)^n \right) = \left(\lim \frac{n^2+1}{2n^2+3}, \lim \left(1+\frac{1}{n}\right)^n \right) = (\frac{1}{2}, e).$$

Limits of functions $f : \mathbb{R}^n \to \mathbb{R}$

Definition (Heine)

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{a} \in \overline{\Omega}$. We say that f has limit b as $\boldsymbol{x}$ tends to $\boldsymbol{a}$, and write it like $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = b$, if for every sequence $(\boldsymbol{a}_n) \subset \Omega$, $a_n \neq \boldsymbol{a}$, such that $\lim \boldsymbol{a}_n = \boldsymbol{a}$ we have $\lim f(\boldsymbol{a}_n) = b$. More precisely,

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = b \Leftrightarrow \forall (\boldsymbol{a}_n), \boldsymbol{a}_n \neq \boldsymbol{a} : \lim \boldsymbol{a}_n = \boldsymbol{a} \Rightarrow \lim f(\boldsymbol{a}_n) = b.$$

Definition (Cauchy)

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\boldsymbol{a} \in \overline{\Omega}$. We say that f has limit b as $\boldsymbol{x}$ tends to $\boldsymbol{a}$, and write it like $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = b$ if

$$\forall_{\varepsilon>0} \quad \exists_{\delta>0} : \forall \boldsymbol{x} \in \Omega, \boldsymbol{x} \neq \boldsymbol{a}, \|\boldsymbol{x} - \boldsymbol{a}\| < \delta \Rightarrow |f(\boldsymbol{x}) - b| < \varepsilon$$

Example (Show that $\lim_{(x,y)\to(0,0)} (2x + y + 1) = 1$)

1. Using Heine's definition. Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (0, 0)$. In that case we know that $\lim x_n = \lim y_n = 0$. Now we just need to show that $\lim f(x_n, y_n) = 1$.

$$\lim f(x_n, y_n) = \lim(2x_n + y - n + 1) = 2\underbrace{\lim x_n}_{=0} + \underbrace{\lim y_n}_{=0} + 1 = 1$$

2. Using Cauchy's definition. For each $\varepsilon > 0$ we must provide a $\delta > 0$ such that $\|(x, y) - (0, 0)\| < \delta \Rightarrow |f(x, y) - 1| < \varepsilon$.

$$\begin{aligned} |f(x,y) - 1| &= |2x + y| \leq 2|x| + |y| = 2\sqrt{x^2} + \sqrt{y^2} \\ &\leq 2\sqrt{x^2 + y^2} + \sqrt{x^2 + y^2} = 3\|(x, y) - (0, 0)\| \end{aligned}$$

Example (cont.)

Finally, given that $|f(x,y)-1| < 3\|(x,y)-(0,0)\|$, we realize that if we set $\delta < \frac{\varepsilon}{3}$ we will have the desired inequality:

$$|f(x,y)-1| \leq 3\|(x,y)-(0,0)\| \leq 3 \times \frac{\varepsilon}{3} = \varepsilon.$$

This shows that $\lim\limits_{(x,y)\to(0,0)} (2x+y+1) = 1.$

Properties of limits

Proposition

Let $f, g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{a} \in \overline{\Omega}$ *such that*

$$b = \lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) =, \qquad c = \lim_{\boldsymbol{x}\to\boldsymbol{a}} g(\boldsymbol{x}).$$

then,

- $\lim_{\boldsymbol{x}\to\boldsymbol{a}} (\alpha f(\boldsymbol{x})) = \alpha \lim_{\boldsymbol{x}\to\boldsymbol{a}} f(x) = \alpha b, \quad \alpha \in \mathbb{R}.$
- $\lim_{\boldsymbol{x}\to\boldsymbol{a}} (f(\boldsymbol{x}) + g(\boldsymbol{x})) = \lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) + \lim_{\boldsymbol{x}\to\boldsymbol{a}} g(\boldsymbol{x}) = b + c.$
- $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(x)g(x) = \left(\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x})\right)\left(\lim_{\boldsymbol{x}\to\boldsymbol{a}} g(\boldsymbol{x})\right) = bc.$
- $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(x)/g(x) = \left(\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x})\right) / \left(\lim_{\boldsymbol{x}\to\boldsymbol{a}} g(\boldsymbol{x})\right) = b/c.$

Properties of limits

Proposition (Limits of composite functions)

Consider $\boldsymbol{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\boldsymbol{g} : B \subset \mathbb{R}^m \to \mathbb{R}^p$ *such that* A, B *are open sets,* $\boldsymbol{f}(A) \subseteq B$, $\boldsymbol{a} \in \overline{A}$, $\boldsymbol{b} \in \overline{B}$. *If there exists* $\lim\limits_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{b}$ *and* $\lim\limits_{\boldsymbol{y}\to\boldsymbol{b}} \boldsymbol{g}(\boldsymbol{y})$ *then we have,*

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} (\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{x}) = \lim_{\boldsymbol{y}\to\boldsymbol{b}} \boldsymbol{g}(\boldsymbol{y}).$$

Example

The limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ cannot be immediately computed because it leads to an indetermination. However, if we consider a new variable $u = x^2 + y^2$ we can rewrite the limit as

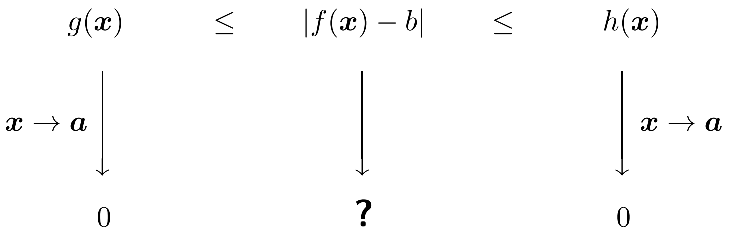
$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0} \frac{\sin u}{u} = 1.$$

Remark

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = b \iff \lim_{\boldsymbol{x}\to\boldsymbol{a}} |f(\boldsymbol{x}) - b| = 0$$

Proposition

Let $g, f, h : \mathbb{R}^n \to \mathbb{R}$ *be defined in a neighborhood* $\boldsymbol{a}$*. If* $\lim_{\boldsymbol{x}\to\boldsymbol{a}} g(\boldsymbol{x}) = \lim_{\boldsymbol{x}\to\boldsymbol{a}} h(\boldsymbol{x}) = b$ *then* $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x})$ *exists and is equal to* b*.*



Some useful inequalities

$$|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$$

$$|a + b| \leq |a| + |b|$$

$$|a - b| \leq |a| + |b|$$

$$||a| - |b|| \leq |a - b|$$

$$|\sin a| \leq 1$$

$$|\cos a| \leq 1$$

$$\ldots$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} =?$$

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \frac{|x||y|}{\sqrt{x^2+y^2}} \leq \frac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$$

So, the previous result applies with $g(x,y)=0$ and $h(x,y)=\sqrt{x^2+y^2}$. Since $|f(x,y)|$ is bounded from bellow and from above by functions that tend to zero as $(x,y)\to 0$, we conclude that the limit under anlysis is in fact zero.

Limits along given sets

Definition

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and consider a set $B \subset \Omega$ such that $\boldsymbol{a} \in \Omega' \cap B'$. The limit of f as $\boldsymbol{x} \to \boldsymbol{a}$ by points of B exists and is equal to b if

$$\forall_{\varepsilon>0} \exists_{\delta>0} : \boldsymbol{x} \in B, \boldsymbol{x} \neq \boldsymbol{a}, \|\boldsymbol{x} - \boldsymbol{a}\| < \delta \Rightarrow |f(\boldsymbol{x}) - b| < \varepsilon$$

Example

Let $B = \{(x, y) \in \mathbb{R}^2 : y = 2x\}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \dfrac{xy^2}{x^2 + y}$. Then

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in B}} f(x, y) = \lim_{\substack{(x,y)\to(0,0)\\y=2x}} f(x, y) = \lim_{x\to 0} f(x, 2x)$$

Limits along sets

Proposition

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *and consider any set* $B \subset \Omega$ *such that* $a \in \Omega` \cap B'$. *Then*

1. *If* $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x})$ *exists and is equal to* b *then, for all* B,

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{a}\\ \boldsymbol{x}\in B}} f(\boldsymbol{x}) = b$$

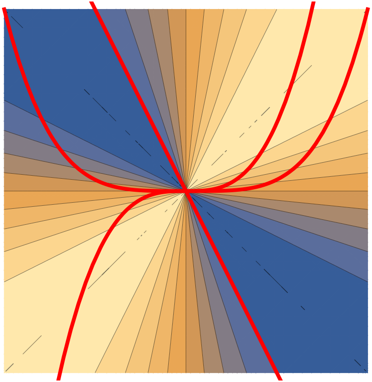
2. *If we there exist sets* B_1, B_2 *such that* $\lim_{\substack{\boldsymbol{x}\to\boldsymbol{a}\\ \boldsymbol{x}\in B_1}} f(\boldsymbol{x}) \neq \lim_{\substack{\boldsymbol{x}\to\boldsymbol{a}\\ \boldsymbol{x}\in B_2}} f(\boldsymbol{x})$, *then* $\lim_{\substack{\boldsymbol{x}\to\boldsymbol{a}\\ \boldsymbol{x}\in B}} f(\boldsymbol{x})$ *does not exist.*

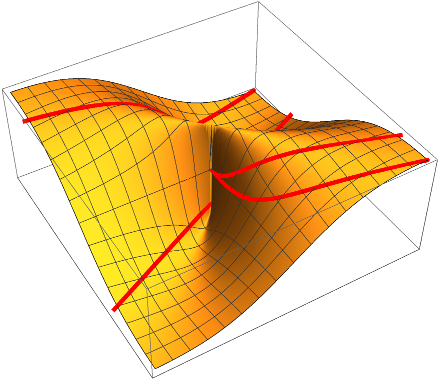
What can go wrong?

Lets considerer $f(x,y)=\dfrac{xy}{x^2+y^2}$ e compute some limits, as $(x,y)\to(0,0)$, along different sets.

- $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=x}}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{x^2}{2x^2}=\frac{1}{2}.$
- $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=2x}}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{2x^2}{5x^2}=\frac{1}{5}.$
- $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=3x}}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{3x^2}{10x^2}=\frac{3}{10}.$
- $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}}\frac{xy}{x^2+y^2}=\lim_{x\to 0}\frac{x^3}{x^2+x^4}=0.$

Since the limit along different sets has different values, **the limit does not exist!**





Proposition

Let $\Omega = B_1 \cup B_2 \cup \cdots \cup B_k$, *for some* $k \in \mathbb{N}$, *with* $B_i \cap B_j = \emptyset, i \neq j$ *and* $\boldsymbol{a} \in B_1' \cap B_2' \cap \cdots B_k'$.

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} f(\boldsymbol{x}) = b \Leftrightarrow \lim_{\substack{\boldsymbol{x} \to \boldsymbol{a} \\ \boldsymbol{x} \in B_i}} f(\boldsymbol{x}) = b, \quad i = 1, \cdots, k.$$

Example

Check that if $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} x^2 + y^2 & , y \geq 0 \\ xy & , y < 0 \end{cases},$$

then we have that $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

Extension to vector functions

Definition (Cauchy)

Let $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{a} \in \Omega'$. We say that $\boldsymbol{f}$ has limit $\boldsymbol{b} \in \mathbb{R}^m$ as $\boldsymbol{x}$ tends to $\boldsymbol{a}$, and write it like $\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{b}$ if

$$\forall_{\varepsilon>0} \quad \exists_{\delta>0} : \forall \boldsymbol{x} \in \Omega, \boldsymbol{x} \neq \boldsymbol{a}, \|\boldsymbol{x} - \boldsymbol{a}\| < \delta \Rightarrow \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{b}\| < \varepsilon$$

Proposition

Let $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{a} \in \Omega'$, such that $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))$. Then the limit $\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{b}$ if and only if

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f_1}(\boldsymbol{x}) = b_1, \quad \lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f_2}(\boldsymbol{x}) = b_2, \ldots, \lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f_m}(\boldsymbol{x}) = b_m,$$

Continuous functions

Definition (Continuity)

Let $\boldsymbol{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\boldsymbol{a} \in \Omega$. Se say that $\boldsymbol{f}$ is continuous at $\boldsymbol{x} = \boldsymbol{a}$ if

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a}).$$

Reasons for not being continuous:

$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x})$ does not exist.

$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x})$ exists but is not equal to $\boldsymbol{f}(\boldsymbol{a})$

Example

$f(x,y) = x^2 + y^2$ is continuous at $(0,0)$ because

$$f(0,0) = 0 = \lim_{(x,y)\to(0,0)} f(x,y).$$

Example

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

is not continuous because $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuous functions: first properties

The definition of a continuous functions is so closely related to the definition of limit that most properties of limits translate directly to continuous functions.

Proposition

Let $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{a} \in \Omega$. If f, g are continuous at $\boldsymbol{x} = \boldsymbol{a}$ then the same is true for

i. αf, $\alpha \in \mathbb{R}$, $f + g$ *and* fg.
ii. f/g, *if* $g(\boldsymbol{a}) \neq 0$.

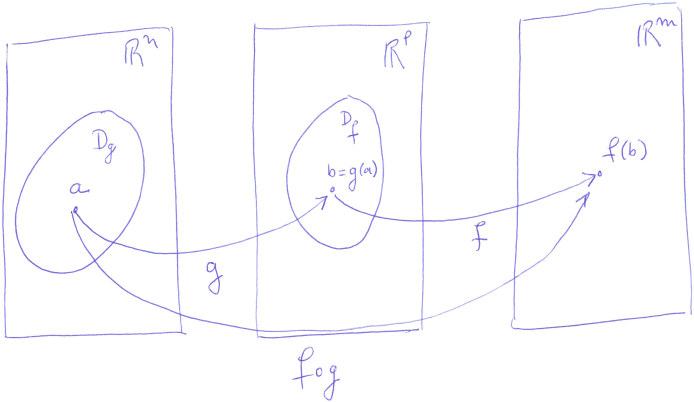
Example

Since $f(x_1, \ldots, x_n) = x_i$ is continuous (check!), any polynomial on the variables $x_1, \ldots, x_n$ is continuous at any point in $\mathbb{R}^n$. For instante, $f(x, y, z) = x^2 z + y^2 + yz$ is a continuous function.

Composition of continuous functions

Proposition

Let $g : D_g \subset \mathbb{R}^n \to \mathbb{R}^p$, $f : D_f \subset \mathbb{R}^p \to \mathbb{R}^m$. If g is continuous at $\boldsymbol{a} \in D_g$ and f is continuous at $\boldsymbol{b} = g(\boldsymbol{a})$, then the composite function $f \circ g : D_g \to \mathbb{R}^m$ is continuous at $\boldsymbol{a}$.



Example

Justify that $f(x,y) = \dfrac{e^{xy+1}}{x^2+y^2+1}$ is continuous everywhere in $\mathbb{R}^2$.

- $xy+1$ and x^2+y^2+1 are polynomials and are therefore continuous.
- e^{xy+1} is a composition of two continuous functions: a polynomial and an exponential, and is therefore continuous.
- Both e^{xy+1} and x^2+y^2+1 are continuous and $x^2+y^2+1 \neq 0$ and so $\dfrac{e^{xy+1}}{x^2+y^2+1}$ is continuous.

Weierstrass' theorem

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a compact set and $f : \Omega \to \mathbb{R}$ a continuous everywhere in Ω. Then f has a global minimum and maximum over the set Ω.

- This result is very important in optimization.
- Is does not give any hint on where to look for the minimum or maximum points.