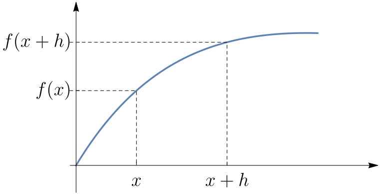
Computing derivatives

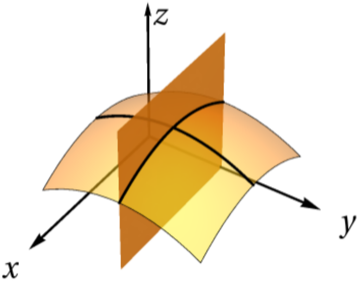
- Derivatives are a powerful tool in studying functions of one variable, providing a simple way of answering questions like: When is the function **increasing**? When is it **decreasing**? Where can it attain a **maximum** or a **minimum**?
- They measure the instantaneous rate of variation, the speed at witch a given function is changing in value.

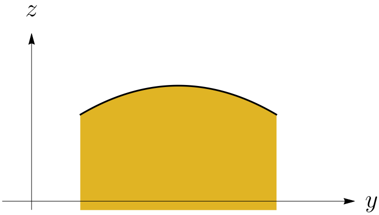


$$f'(x) = \lim_{h\to 0} \frac{\overbrace{f(x+h) - f(x)}^{\text{Increase in } f}}{\underbrace{h}_{\text{Increase in } x}}$$

Partial derivatives

- When f depends on more variables, it can be increasing in one direction but decreasing in another. The discussion of monotonicity must include a given direction.





$$f(x,y) = 4-(x-2)^2-(y-2)^2 \qquad \psi(y) = f(2,y) = 4-(y-2)^2$$

Partial derivatives

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ and suppose we want to evaluate the increase in the value of f when we move away from (x, y), in the x-direction or in the y - direction. If we move h, the mean rate of variation is given by

$$\frac{\overbrace{f(x+h,y)-f(x,y)}^{\text{Increase in } f}}{\underbrace{h}_{\text{Increase in } x}} \qquad \frac{\overbrace{f(x,y+h)-f(x,y)}^{\text{Increase in } f}}{\underbrace{h}_{\text{Increase in } x}}$$

The instantaneous rate of variation with respect to x is defined as

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

The instantaneous rate of variation with respect to y is defined as

$$\lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

- The partial derivative with respect to x is defined as the instantaneous rate of variation with respect to x and is denoted by

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

- The partial derivative with respect to y is defined as the instantaneous rate of variation with respect to y and is denoted by

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Definition (partial derivative)

Let $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}$ and $a\in int(\Omega)$. The partial derivative of f with respect to x_i at a is denoted by $f'_{x_i}(a)$ or $\frac{\partial f}{\partial x_i}(a)$ and is defined as

$$f'_{x_i}(a)=\lim_{h\to 0}\frac{f(x_1,\cdots,\overbrace{x_i+h}^{i^{th}\text{ position}},\cdots,x_n)-f(x_1,\cdots,x_i,\cdots,x_n)}{h}$$

If we fix $x_k=a_k$ for $k\neq i$, we can define the real function $x\mapsto\psi_i(x)=f(a_1,\cdots,\underbrace{x}_{i^{th}},\cdots,a_n)$, and we have that

$$\frac{\partial f}{\partial x_i}(a_1,\cdots,a_n)=\psi'_i(a_i)$$

Example

Consider $f(x_1, x_2) = x_1^2 + x_2^2$. If we want to compute the partial derivatives at $(1, 2)$ we need only consider the partial functions

$$\psi_1(x_1) = f(x_1, 2) = x_1^2 + 4, \qquad \psi_2(x_2) = f(1, x_2) = 1 + x_2^2$$

$$\psi_1'(x_1) = 2x_1, \qquad \psi_2'(x_2) = 2x_2$$

$$\frac{\partial f}{\partial x_1}(1,2) = \psi_1'(1) = 2, \qquad \frac{\partial f}{\partial x_2}(1,2) = \psi_2'(2) = 4$$

From the practical point of view:

- We compute f'_{x_1} considering x_2 as a constant and computing the usual derivative using x_1 as the variable.
- We compute f'_{x_2} considering x_1 as a constant and computing the usual derivative using x_2 as the variable.

Example

Let $f(x,y,z) = x^2y + y^2z + \sin(x^2 + y^3 + z^4)$. Then

$$\frac{\partial f}{\partial x} = (x^2y)'_x + (y^2z)'_x + (x^2 + y^3 + z^4)'_x \cos(x^2 + y^3 + z^4)$$
$$= 2xy + 2x\cos(x^2 + y^3 + z^4)$$
$$\frac{\partial f}{\partial y} = (x^2y)'_y + (y^2z)'_y + (x^2 + y^3 + z^4)'_y \cos(x^2 + y^3 + z^4)$$
$$= x^2 + 2yz + 3y^2\cos(x^2 + y^3 + z^4)$$
$$\frac{\partial f}{\partial z} = (x^2y)'_z + (y^2z)'_z + (x^2 + y^3 + z^4)'_z \cos(x^2 + y^3 + z^4)$$
$$= y^2 + 4z^3\cos(x^2 + y^3 + z^4)$$

Example

$$f(x,y) = \begin{cases} 2y + \dfrac{x^2y^2}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,x = y = 0 \end{cases}$$

- If we want to compute the partial derivatives at $(0,0)$ we **must use the definition**, because f is defined by more then one expression in any neighborhood of $(0,0)$.
- If we want to compute the partial derivatives at any other point $(x,y) \neq (0,0)$ we can use the usual rules for derivatives.

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{2h-0}{h} = 2$$

Example (cont.)

For any $(x,y) \neq (0,0)$ we have that

$$\frac{\partial f}{\partial x} = 0 + \frac{(x^2y^2)'_x(x^2+y^2) - (x^2y^2)(x^2+y^2)'_x}{(x^2+y^2)^2}$$

$$= \frac{2xy^2(x^2+y^2) - x^2y^2 2x}{(x^2+y^2)^2)} = \frac{2xy^4}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = (2y)'_y + \frac{(x^2y^2)'_y(x^2+y^2) - (x^2y^2)(x^2+y^2)'_y}{(x^2+y^2)^2}$$

$$= 2 + \frac{2x^2y(x^2+y^2) - x^2y^2 2y}{(x^2+y^2)^2)} = 2 + \frac{2x^4y}{(x^2+y^2)^2}$$

We can substitute x, y in these expressions and get the values of the partial derivatives, in any point except $(0,0)$.

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Example (cont.)

Finalmente,

$$\frac{\partial f}{\partial x}(x,y)=\begin{cases}\dfrac{2xy^4}{(x^2+y^2)^2} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial y}(x,y)=\begin{cases}2+\dfrac{2x^4y}{(x^2+y^2)^2} & ,(x,y)\neq(0,0)\\ 2 & ,(x,y)=(0,0)\end{cases}$$

- Unlike what happens for functions of one variable, the existence of partial derivatives does not imply the continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$. For example

$$f(x,y) = \begin{cases} 1 & , \text{se } xy = 0 \\ 0 & , \text{se } xy \neq 0 \end{cases}$$

  is obviously discontinuous at $(0,0)$ but both partial derivatives exist and are equal to zero.
- This fact indicates that the concept of partial derivative is probably not the best extension of the concept of differentiability.

Directional derivatives

The partial derivatives measure the rate of variation of a given function in the direction of the coordinate axis. One possible extension is to assess the rate of variation in the direction of a given vector $\boldsymbol{v}$.

Definition

Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}$, $\boldsymbol{a} \in \Omega$ and $\boldsymbol{v} \in \mathbb{R} \setminus \{0\}$. The **derivative of** f **along** $\boldsymbol{v}$ is defined by the limit

$$\partial_{\boldsymbol{v}} f(\boldsymbol{a}) = \lim_{h \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{v}) - f(\boldsymbol{a})}{h}$$

$$= \lim_{h \to 0} \frac{f(a_1 + hv_1, \cdots, a_n + hv_n) - f(a_1, \cdots, a_n)}{h}.$$

If $\|\boldsymbol{v}\| = 1$ we call this the **directional derivative** of f along $\boldsymbol{v}$.

Example

Compute all directional derivatives of f at $(0,0)$, where

$$f(x,y)=\begin{cases}\dfrac{xy^2}{x^2+y^4} & ,(x,y)\neq(0,0)\\ 0 & ,x=y=0\end{cases}$$

$$\begin{aligned}\partial_{\boldsymbol{v}} f(0,0) &= \lim_{h\to 0}\frac{f(0+hv_1,0+hv_2)-f(0,0)}{h}\\ &= \lim_{h\to 0}\frac{\dfrac{hv_1(hv_2)^2}{(hv_1)^2+(hv_2)^4}-0}{h} = \lim_{h\to 0}\frac{v_1v_2^2}{v_1^2+h^2v_2^4}\\ &= \begin{cases}0, & v_1=0\\ \frac{v_2^2}{v_1}, & v_1\neq 0\end{cases}\end{aligned}$$

Example (cont.)

If easy to check that f is not continuous at $(0,0)$. The existence of all directional derivatives is not enough to guarantee that a function is continuous.

Remark

The partial derivatives are a particular case of directional derivatives, in fact

$$\frac{\partial f}{\partial x_i} = \partial_{e_i} f$$

For example, if $f : \mathbb{R}^2 \to \mathbb{R}$*, we have that*

$$\frac{\partial f}{\partial x}(x,y) = \partial_{(1,0)} f(x,y), \qquad \frac{\partial f}{\partial y}(x,y) = \partial_{(0,1)} f(x,y)$$

Differentiability

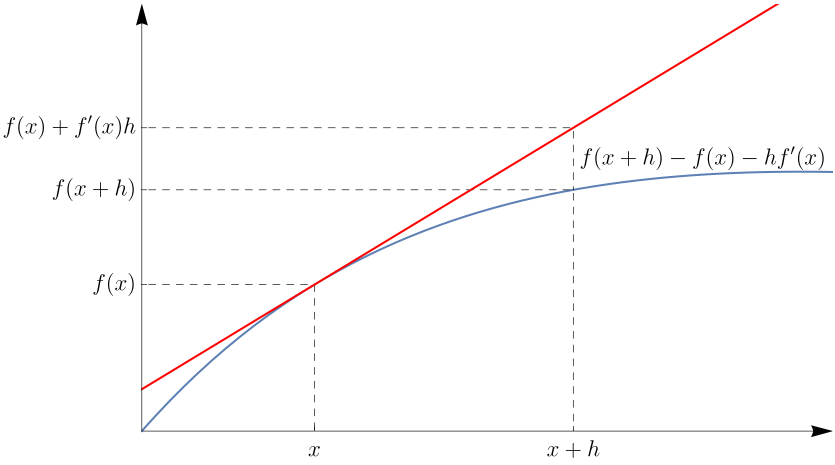
For functions of one variable we say that f has derivative at a or that f is **differentiable** at a if the following limit exists:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \stackrel{\text{def}}{=} f'(a)$$

This is equivalent to say that there exists a constant, that we call $f'(a)$, such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0,$$

This means that the increments of f are "well" explained by the linear function $Df(h) = f'(a)h$.


$f(x) + f'(x)h$
$f(x+h) - f(x) - hf'(x)$
$f(x+h)$
$f(x)$
x
$x+h$

Differentiability

Definition

Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}$ and $\boldsymbol{a} \in \Omega$ such that $B_\delta(\boldsymbol{a}) \subset \Omega$. We say that f is **differentiable** at $\boldsymbol{a}$ if there exists a linear aplication $D_1 : \mathbb{R}^n \to \mathbb{R}$ such that, for all $\boldsymbol{h}$ with $\|\boldsymbol{h}\| < \delta$ such that

$$f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) = D_1(\boldsymbol{h}) + \mathcal{E}(\boldsymbol{h})\|\boldsymbol{h}\|,$$

where $\lim_{\boldsymbol{h}\to 0} \mathcal{E}(\boldsymbol{h}) = 0$.

Remark

We can say, equivalentlly, that f is differentiable at $\boldsymbol{a}$ iif

$$\lim_{\boldsymbol{h}\to 0} \frac{f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - D_1(\boldsymbol{h})}{\|\boldsymbol{h}\|} = 0$$

Proposition

If f is differentiable at $\boldsymbol{a}$ all directional derivatives exist at $\boldsymbol{a}$, and we have

$$D_1(\boldsymbol{v}) = \partial_{\boldsymbol{v}} f(\boldsymbol{a}).$$

In particular, all partial derivatives exist at $\boldsymbol{a}$ and

$$D_1(\boldsymbol{h}) = h_1 \frac{\partial f}{\partial x_1}(\boldsymbol{a}) + \cdots + h_n \frac{\partial f}{\partial x_n}(\boldsymbol{a})$$

Definition

The linear application introduced above is called the **first order differential** of f at $\boldsymbol{a}$, and is denoted by

$$Df(\boldsymbol{a})(\boldsymbol{h}) = h_1 \frac{\partial f}{\partial x_1}(\boldsymbol{a}) + \cdots + h_n \frac{\partial f}{\partial x_n}(\boldsymbol{a})$$

Differentiability

Considering the previous results we can rewrite the definition of Differentiability as follows:

Definition (Differentiability)

Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}$ and $\boldsymbol{a} \in \Omega$. Se say that f is **differentiable** at $\boldsymbol{a}$ if

$$\lim_{\boldsymbol{h}\to 0} \frac{f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - \displaystyle\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\boldsymbol{a})}{\|\boldsymbol{h}\|} = 0.$$

Example

The function $f(x,y) = xy$ is differentiable at any point $(a_1, a_2) \in \mathbb{R}^2$. To check this we only need to evaluate the limit

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - h_1 f'_x(\boldsymbol{a}) - h_2 f'_y(\boldsymbol{a})}{\sqrt{h_1^2+h_2^2}} =$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{(a_1+h_1)(a_2+h_2) - a_1a_2 - a_2h_1 - a_1h_2}{\sqrt{h_1^2+h_2^2}} =$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{\cancel{a_1a_2} + \cancel{a_1h_2} + \cancel{h_1a_2} + h_1h_2 - \cancel{a_1a_2} - \cancel{a_2h_1} - \cancel{a_1h_2}}{\sqrt{h_1^2+h_2^2}} =$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{h_1h_2}{\sqrt{h_1^2+h_2^2}} := L$$

Example (cont.)

Now, two situations can occur:

- If L exists and is equal to zero, f is differentiable.
- If L does not exist or exists but it is not zero, the function in not differentiable.

In this case we can see that

$$\left|\frac{h_1h_2}{\sqrt{h_1^2+h_2^2}}\right| = \frac{|h_1||h_2|}{\sqrt{h_1^2+h_1^2}} \leq \frac{\sqrt{h_1^2+h_1^2}\cdot\sqrt{h_1^2+h_1^2}}{\sqrt{h_1^2+h_1^2}}$$

$$= \sqrt{h_1^2+h_1^2} \overset{(h_1,h_2\to 0)}{\longrightarrow} 0$$

And so f is differentiable at any point $(a_1, a_2) \in \mathbb{R}^2$.

Proposition

Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}$ and $\boldsymbol{a} \in \Omega$. If all partial derivatives exist at $\boldsymbol{a}$ and are continuous in a neighborhood of $\boldsymbol{a}$, then f is differentiable at $\boldsymbol{a}$.

Proposition

Let $\Omega \subset \mathbb{R}^n$ be an open set, $f : \Omega \to \mathbb{R}$ and $\boldsymbol{a} \in \Omega$. If f is differentiable at $\boldsymbol{a}$ then

i. *f is continuous at $\boldsymbol{a}$.*
ii. *all partial derivatives exist at $\boldsymbol{a}$.*
iii. *All directional derivatives exist at $\boldsymbol{a}$.*
iv. $\partial_{\boldsymbol{v}} f(\boldsymbol{a}) = h_1 \frac{\partial f}{\partial x_1}(\boldsymbol{a}) + \cdots + h_n \frac{\partial f}{\partial x_n}(\boldsymbol{a})$.