Models in Finance - Class 18

Master in Actuarial Science

João Guerra

ISEG

Black-Scholes model - PDE approach

- idea: use Itô's formula to derive an expression for the price of the derivative as a function $f(S_t)$ of S_t and then construct a risk-free portfolio.
- By Itô's formula:

$$df(t, S_t) = \frac{\partial f}{\partial t}(t, S_t)\, dt + \frac{\partial f}{\partial S_t}(t, S_t)\, dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(t, S_t)\,(dS_t)^2 . \tag{1}$$

- Recall that $dS_t = S_t(\mu dt + \sigma dZ_t)$ and therefore

$$\begin{aligned}(dS_t)^2 &= S_t^2\left[\mu^2 (dt)^2 + \sigma^2 (dZ_t)^2 + 2\mu\sigma dt dZ_t\right]\\ &= \sigma^2 S_t^2 dt\end{aligned}$$

(why?)

PDE approach

- Therefore:

$$df(t,S_t) = \frac{\partial f}{\partial t}(t,S_t)\,dt + \frac{\partial f}{\partial S_t}(t,S_t)\left[S_t\left(\mu dt + \sigma dZ_t\right)\right]$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial S_t^2}(t,S_t)\,\sigma^2 S_t^2 dt$$
$$= \left[\frac{\partial f}{\partial t}(t,S_t) + \mu S_t \frac{\partial f}{\partial S_t}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(t,S_t)\right] dt \quad (2)$$
$$+ \sigma S_t \frac{\partial f}{\partial S_t}(t,S_t)\,dZ_t. \quad (3)$$

PDE approach

- At time t with $0 \leq t < T$, consider you hold the portfolio:
- -1 derivative $+ \frac{\partial f}{\partial S_t}(t,S_t)$ shares
- Let $V(t,S_t)$ be the value of this portfolio:

$$V(t,S_t) = -f(t,S_t) + \frac{\partial f}{\partial S_t}(t,S_t)\,S_t.$$

- The variation of the portfolio value over the period $(t, t+dt]$ is (by Eq. (2) and (3))

$$-df(t,S_t) + \frac{\partial f}{\partial S_t}(t,S_t)\,dS_t$$
$$= -\left(\frac{\partial f}{\partial t}(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(t,S_t)\right) dt \quad (4)$$

PDE approach

- $-df(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t)\, dS_t$ involves dt but not $dZ_t \Longrightarrow$ instantaneous investment gain in $(t, t+dt]$ is risk-free.
- arbitrage-free market $\Longrightarrow$risk-free rate $= r \Longrightarrow$

$$-df(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t)\, dS_t = rV(t, S_t)\, dt. \qquad (5)$$

- By (4) and (5), we have:

$$\left(\frac{\partial f}{\partial t}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2}(t, S_t)\right) dt = -rV(t, S_t)\, dt$$
$$= -r\left(-f(t, S_t) + \frac{\partial f}{\partial S_t}(t, S_t)\, S_t\right) dt$$

and therefore (substituting $S_t = s$)

$$\frac{\partial f}{\partial t}(t, s) + rs\frac{\partial f}{\partial s}(t, s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2}(t, s) = rf(t, s). \qquad (6)$$

- This is the Black-Scholes PDE (partial differential equation).

PDE approach

- The value of the derivative $f(t, S_t)$ is obtained by solving the B-S PDE with appropriate boundary conditions, which are for the call and put:

$$f(T, s) = \max\{s - K, 0\} \quad \text{for the call,}$$
$$f(T, s) = \max\{K - s, 0\} \quad \text{for the put.}$$

- We can try out the solutions given in the proposition:

$$f(t, S_t) = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \quad \text{for the call,} \qquad (7)$$
$$f(t, S_t) = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1) \quad \text{for the put,} \qquad (8)$$

and find that they satisfy the PDE and the appropriate boundary conditions.

PDE approach

- Exercise: A forward contract is arranged where an investor agrees to buy a share at time T for an amount K. It is proposed that the fair price of this contract is

$$f(t, S_t) = S_t - Ke^{-r(T-t)}.$$

Show that this:
(i) Satisfies the appropriate boundary condition.
(ii) Satisfies the Black-Scholes PDE.

The martingale approach

- In the binomial model, we proved that the value of a derivative could be expressed by:

$$V_t = e^{-r(T-t)} E_Q[X|\mathcal{F}_t],$$

where X is the value of the derivative at maturity T and Q is the equivalent martingale measure (or risk neutral measure).
- In continuous time, this result can be generalized as:

Proposition: Let X be any derivative payment contingent on $\mathcal{F}_T$, payable at T. Then the value of this derivative at time $t < T$ is

$$V_t = e^{-r(T-t)} E_Q[X|\mathcal{F}_t]. \quad (9)$$

The martingale approach

Sketch of the Proof: We can use the 5-step method as in the binomial model case:

- Step 1: Establish the unique equivalent measure Q under which $D_t = e^{-rt}S_t$ is a martingale.
  It can be shown that this measure exists, is unique and under Q, we have $D_t = D_0 \exp(\sigma\widetilde{Z}_t - \frac{1}{2}\sigma^2 t)$, where $\widetilde{Z}_t$ is a Q-Brownian motion.
- Step 2: Define $V_t = e^{-r(T-t)}E_Q\left[X|\mathcal{F}_t\right]$. We propose this as the "fair"price of the derivative.
- Step 3: Let $E_t = e^{-rt}V_t = e^{-rT}E_Q\left[X|\mathcal{F}_t\right]$. It can be shown that under Q, E_t is a martingale.
- Step 4: By the martingale representation theorem (MRT) there exists a previsible process ϕ_t (i.e. ϕ_t is $\mathcal{F}_{t^-}$-measurable) such that:

$$dE_t = \phi_t dD_t.$$

The martingale approach

- Step 5: Let $\psi_t = E_t - \phi_t D_t$. Suppose that at time t, we hold the portfolio:

$$\begin{aligned}&\phi_t \text{ units of asset } S_t,\\&\psi_t \text{ units of cash account } B_t.\end{aligned}$$

  At time t, the porfolio has value

$$\begin{aligned}\phi_t S_t + \psi_t B_t &= e^{rt}\left(\phi_t D_t + \psi_t\right) = e^{rt}E_t\\&= V_t.\end{aligned}$$

  At time $t + dt$:

$$\begin{aligned}\phi_t S_{t+dt} + \psi_t B_{t+dt} &= e^{r(t+dt)}\left(\phi_t D_{t+dt} + \psi_t\right)\\&= e^{r(t+dt)}\left(\phi_t D_t + \phi_t dD_t + \psi_t\right)\\&= e^{r(t+dt)}\left(E_t + dE_t\right)\\&= e^{r(t+dt)}E_{t+dt} = V_{t+dt}.\end{aligned}$$

The martingale approach

- Step 5 (cont.): Therefore:

$$V_{t+dt} - V_t = dV_t = \phi_t dS_t + \psi_t dB_t$$

and the hedging strategy (ϕ_t, ψ_t) is self-financing.
Moreover,

$$V_T = E_Q\left[X|\mathcal{F}_T\right].$$

So, the hedging strategy is a replicating portfolio and
$V_t = e^{-r(T-t)}E_Q\left[X|\mathcal{F}_t\right]$ is the fair price of the derivative at time t.

Delta hedging and martingale approach

- How to determine ϕ_t of the replicating portfolio?
- We can evaluate the price of the derivative $V_t = e^{-r(T-t)}E_Q\left[X|\mathcal{F}_t\right]$ using a formula (like the B-S formula) or numerical techniques.
- Then

$$\phi_t = \frac{\partial V}{\partial s}(t, S_t). \quad (10)$$

- ϕ_t is called the Delta of the derivative:

$$\Delta = \frac{\partial V}{\partial s}(t, S_t). \quad (11)$$

Delta hedging and martingale approach

If:

- we start at time 0 with V_0 invested in cash and shares,
- we follow a self-financing portfolio strategy,
- we continually rebalance the portfolio to hold exactly $\phi_t = \Delta = \frac{\partial V}{\partial s}(t, S_t)$ units of S_t with the rest in cash,

then we will precisely replicate the derivative payoff.

Example: B-S formula for a call

- Let $X = \max\{S_T - K, 0\}$. Then:

$$V_t = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \quad (12)$$

where: $d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, $d_2 = d_1 - \sigma\sqrt{T-t}$ and $\Phi(z)$ is the cumulative distribution function of the standard normal distribution.

Example: B-S formula for a call

Proof:

- Given the information $\mathcal{F}_t$, then under Q, we have:

$$S_T = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\widetilde{Z}_T - \widetilde{Z}_t\right)\right]. \qquad (13)$$

Then

$$V_t = e^{-r(T-t)} E_Q\left[\max\left\{S_T - K, 0\right\} | \mathcal{F}_t\right]$$
$$= e^{-r(T-t)}$$
$$\times E_Q\left[\max\left\{S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(\widetilde{Z}_T - \widetilde{Z}_t\right)\right] - K, 0\right\} | \mathcal{F}_t\right]$$
$$= E_Q\left[\max\left\{e^{\alpha+\beta U} - e^{\alpha+\beta u}, 0\right\}\right],$$

where $\alpha = \log(S_t) - \frac{1}{2}\sigma^2(T-t)$, $\beta = \sigma\sqrt{T-t}$, $U \sim N(0,1)$ under Q and $u = \left[\log\left(Ke^{-r(T-t)}\right) - \alpha\right]/\beta$.

Example: B-S formula for a call

Proof:

- Therefore (with $\phi(x)$ the density of the $N(0,1)$ distribution):

$$V_t = e^{\alpha+\beta u} \int_u^\infty \left(e^{\beta(x-u)} - 1\right)\phi(x)\,dx$$
$$= e^{\alpha} \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{\beta x - \frac{1}{2}x^2} dx - e^{\alpha+\beta u}\Phi(-u)$$
$$= e^{\alpha + \frac{1}{2}\beta^2} \int_u^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\beta)^2} dx - e^{\alpha+\beta u}\Phi(-u)$$
$$= e^{\alpha+\frac{1}{2}\beta^2}\Phi(\beta - u) - e^{\alpha+\beta u}\Phi(-u) = \ldots$$
$$= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2).$$

- Exercise: Prove the B-S formula for the put option, using the same technique.