Models in Finance - Class 19

Master in Actuarial Science

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B-S model: replication of European call

- Discounted values: $E_t = e^{-rt} V_t$ and $D_t = e^{-rt} S_t$ are both martingales under Q.
- Moreover, we know that under Q,

$$dS_t = S_t \left(rdt + \sigma d\widetilde{Z}_t \right), \tag{1}$$

$$dD_t = \sigma D_t d\widetilde{Z}_t, \tag{2}$$

$$\begin{aligned} dE_t &= -re^{-rt} V_t dt + e^{-rt} dV_t \\ &= e^{-rt} \left(-rV_t dt + dV_t \right). \end{aligned} \tag{3}$$

B-S model: replication of European call

- By Ito's formula:

$$\begin{aligned} dV_t &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial s}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial s^2}(dS_t)^2 \\ &= \left[\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial s^2}\right]dt + \\ &+ \sigma\frac{\partial V}{\partial s}S_t d\widetilde{Z}_t \\ &= \left[\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial s^2}\right]dt + \\ &+ \frac{\partial V}{\partial s}e^{rt}dD_t. \end{aligned}$$

B-S model: replication of European call

- Using Eq. (3) we obtain:

$$dE_t = e^{-rt}\left(-rV_t + \frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial s^2}\right)dt \quad (4)$$

$$+ \frac{\partial V}{\partial s}dD_t. \quad (5)$$

- Since E_t and D_t are both martingales under Q, by the MRT, exists previsible process ϕ_t such that

$$dE_t = \phi_t dD_t = \sigma\phi_t D_t d\widetilde{Z}_t \quad (6)$$

- Comparing Eqs (4)-(5) with Eq. (6), we have that:

$$\phi_t = \frac{\partial V}{\partial s}, \quad (7)$$

$$rV_t = \frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial s^2}. \quad (8)$$

B-S model: replication of European call

- The last PDE equation is the Black-Scholes PDE.
- We can show that:
$$\phi_t = \frac{\partial V}{\partial s} = \Phi\left(d_1\right) \tag{9}$$
- The martingale approach has provided an alternative derivation of the B.-S. PDE and Eq. (9) gives explicit formula for ϕ_t of the replicating portfolio (is equal to the Delta Δ).

Advantages of the martingale approach

- The martingale approach is much more clear in the process of pricing derivatives, comparing to the PDE approach.
- Under the PDE approach we derived a PDE and had to "guess" the solution for a given set of boundary conditions.
- Under the martingale approach we have an expectation which can be evaluated explicitly in some cases and in a straightforward numerical way in other cases.
- The martingale approach also gives us the replicating strategy for the derivative.
- The martingale approach can be applied to any $\mathcal{F}_T$ -measurable derivative payment, including path-dependent options (for example, Asian options), whereas the PDE approach, in general, cannot.

Risk neutral pricing

- Exercise: You are trying to replicate a 6-month European call option with strike price 500, which you purchased 4 months ago. If $r = 0.05$, $\sigma = 0.2$, and the current share price is 475, what portfolio should you be holding (assuming no dividends) ?
- The martingale approach is also known as risk-neutral pricing. The measure Q is commonly called the risk-neutral measure. However, Q is also referred to as the equivalent martingale measure because the discounted prices D_t and E_t are martingales under Q.

State price deflator approach

- Recall that:

$$dS_t = S_t \left(\mu dt + \sigma dZ_t \right), \text{ under } P, \tag{10}$$

$$dS_t = S_t \left(rdt + \sigma d\widetilde{Z}_t \right), \text{ under } Q, \tag{11}$$

where

$$d\widetilde{Z}_t = dZ_t + \gamma dt \tag{12}$$

and

$$\gamma = \frac{\mu - r}{\sigma}. \tag{13}$$

- A corollary to the Cameron-Martin-Girsanov theorem states that there exists a process η_t such that for a payoff X we have:

$$E_Q \left[X | \mathcal{F}_t \right] = E_P \left[\frac{\eta_T}{\eta_t} X | \mathcal{F}_t \right],$$

where (in this case):

$$\eta_t = e^{-\gamma Z_t - \frac{1}{2}\gamma^2 t} \tag{14}$$

State price deflator approach

- Define
$$A_t = e^{-rt}\eta_t. \quad (15)$$
- The price of the derivative is:
$$V_t = e^{-r(T-t)} E_Q \left[X | \mathcal{F}_t \right] = e^{-r(T-t)} E_P \left[\frac{\eta_T}{\eta_t} X | \mathcal{F}_t \right]$$
$$= \frac{E_P \left[A_T X | \mathcal{F}_t \right]}{A_t}. \quad (16)$$
- A_t is called a state-price deflator (also deflator; state-price density; pricing kernel; or stochastic discount factor).

The B-S model with dividends

- Suppose that dividends are payable continuously at the constant rate of q p.a.: that is, the dividend payable over the interval $[t, t + dt]$ is $qS_t dt$.
- Suppose that S_t is subject to the same SDE:
$$dS_t = S_t \left(\mu dt + \sigma dZ_t \right), \text{ under } P.$$
- Let $\widetilde{S}_t$ be the value of an investment of $\widetilde{S}_0 = S_0$ at time 0 in the underlying asset assuming that all dividends are reinvested in the same asset at the time of payment of the dividend.
- $\frac{\widetilde{S}_t}{\widetilde{S}_0}$ described as the total return on the asset from time 0 to time t.

The B-S model with dividends

- $\widetilde{S}_t$ is the tradable asset and not S_t in the following sense: If we pay S_0 at time 0 for the asset then we are buying the right to future dividends as well as future growth of the capital.
- It is straightforward to see that the SDE for $\widetilde{S}_t$ is:

$$d\widetilde{S}_t = \widetilde{S}_t \left[(\mu + q)\, dt + \sigma dZ_t \right], \text{ under } P. \tag{17}$$

Solving the SDE we have (geometric Bm):

$$\widetilde{S}_t = \widetilde{S}_0 \exp \left[\left(\mu + q - \frac{1}{2}\sigma^2 \right) t + \sigma Z_t \right]. \tag{18}$$

The B-S model with dividends

- Denote the value at time t of an European call option on the dividend paying share by $f(t, S_t)$. Then we have:

Proposition: (Garman-Kohlhagen formula):

$$f(t, S_t) = S_t e^{-q(T-t)} \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \tag{19}$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{20}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{21}$$

For a put option, we have

$$f(t, S_t) = K e^{-r(T-t)} \Phi(-d_2) - S_t e^{-q(T-t)} \Phi(-d_1). \tag{22}$$

The B-S model with dividends

- These formulas can be derived by the PDE approach and by the martingale approach (as in the non-dividend-paying case) - see core reading (homework).
- The B.-S. PDE is (substituting $S_t = s$)

$$\frac{\partial f}{\partial t} + (r - q)\, s \frac{\partial f}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} = rf(t, s). \tag{23}$$

- In the martingale approach, the basic SDE for $\widetilde{S}_t$ under Q is (see core reading)

$$d\widetilde{S}_t = \widetilde{S}_t \left[(r - q)\, dt + \sigma d\widetilde{Z}_t \right] \quad \text{under } Q. \tag{24}$$

The B-S model with dividends

- In the martingale approach, we have (see the 5 step method for this case in the core reading)

$$V_t = e^{-r(T-t)} E_Q \left[X | \mathcal{F}_t \right].$$

- By the MRT there exists a previsible process $\widetilde{\phi}_t$ such that:

$$dE_t = \widetilde{\phi}_t d\widetilde{D}_t,$$

  and $\psi_t = E_t - \widetilde{\phi}_t \widetilde{D}_t$.
- $\widetilde{\phi}_t$ units of $\widetilde{S}_t$ is equivalent to $\phi_t = e^{qt}\widetilde{\phi}_t$ units of S_t. At time t, the replicating portfolio is $\left(\widetilde{\phi}_t, \psi_t\right)$ or (ϕ_t, ψ_t) if we consider S_t instead of $\widetilde{S}_t$.
- See more detail on the dividend-paying case of the Black-Scholes model in the Core Reading (homework).