University of Lisbon – ISEG

Departament of Mathematics

Probability Theory and Stochastic Processes

EXAM January 18, 2017

Time limit: 2 hours

Each question: 2.5 points

(1) Consider the probability space $(\mathbb{R}, \mathcal{P}, \delta_a)$, where δ_a is the Dirac measure on $\mathbb{R}$ at $a = 2$, and a random variable $X(x) = \sqrt{|x|}$.

(a) Find the distribution and characteristic functions of X.

(b) Write an example of a random variable Y with the same distribution of X.

(2) For each $n \in \mathbb{N}$ consider a random variable X_n with distribution function

$$F_n(x) = \begin{cases} 0, & x \leq 0 \\ nx, & 0 < x \leq \frac{1}{n} \\ 1, & x > \frac{1}{n}. \end{cases}$$

Find the limit in distribution of X_n as $n \to +\infty$.

(3) Consider a homogeneous Markov chain with transition matrix given by

$$P = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Classify the states of the chain.

(b) Determine the period of each state.

(c) If possible, find the stationary distributions and the mean recurrence time of each state.

(4) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_1, X_2, \dots$ a sequence of iid random variables with distribution

$$P(X_n = 1) = \frac{1}{2} \quad \text{and} \quad P(X_n = -1) = \frac{1}{2}.$$

Consider the stopping time

$$\tau = \min\{n \in \mathbb{N} \colon X_n = 1\}$$

with respect to the filtration $\sigma(X_1, \dots, X_n)$.

(a) Decide if $X_{\tau \wedge n}$ is a martingale, where $\tau \wedge n = \min\{\tau, n\}$.

(b) Let $S_n = \sum_{i=1}^{n} 2^i X_i$. Compute $E(S_{\tau-1})$.