Master in Actuarial Science
Risk Theory

Exam 1, 06/06/2016
Time allowed: Three hours

Instructions:

1. This paper contains 7 questions and comprises 3 pages including the title page;
2. Enter all requested details on the cover sheet;
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so;
4. Number the pages of the paper where you are going to write your answers;
5. Attempt all 7 questions;
6. Begin your answer to each of the 7 questions on a new page;
7. Marks are shown in brackets. Total marks: 200;
8. Show calculations where appropriate;
9. An approved calculator may be used. No mobile phones are permitted;
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider the (a, b, i), $i = 0, 1$, classes of distributions.

   (a) Do the values $a = 1/2$ and $b = 4/3$ lead to a legitimate distribution of class $(a, b, 0)$? Explain. [5]

   (b) Consider the distribution
   $$f(x) = 0.1(0.9)^{x-1}\,, \quad x = 1, 2, \ldots$$
   Find the values a and b such that $f(x)$ is a member of either $(a, b, 0)$ or $(a, b, 1)$ class. Identify the class. [5]

   (c) Use the probability generating function and the iteration property of the expected value to show that a Poisson mixed with a Gamma distribution leads to a negative binomial. [10]

2. Consider an existing ordinary deductible, say d, for a certain portfolio of risks. The number of losses occurring is Poisson with parameter λ and single claim amount distribution is $F_X(.)$, so that aggregate losses are compound Poisson.

   (a) With such a deductible put in force, the aggregate claim payments incurred by the insurer is (still) compound Poisson. Discuss appropriately, identify parameters, writing in terms of the original λ and F_X. [15]

   (b) Consider that single claim amount distribution is Exponential with mean 10, $\lambda = 0.1$ and deductible $d = 2.5$. Calculate the expected aggregate loss incurred by the insurer. [10]

3. Consider two independent compound Poisson random variables S_1 and S_2, with Poisson parameters λ_1 and λ_2, and single claim probability functions f_1 and f_2, respectively. Let $\lambda_1 = 1$, $\lambda_2 = 1/2$, $f_1(1) = f_1(2) = 1/2$, $f_2(1) = 1$, and $S = S_1 + S_2$.

   (a) Use generating functions to show that the probability function of S is compound Poisson with parameter $\lambda = 3/2$ and single claim amount probability function associated to S, say f_X, is given by a weighted average of f_1 and f_2. [15]

   (b) Compute the convolution $f_X^{*3}(x)$, $x = 0, 1, 2, 3, 4, 5, 6$. [15]

   (c) Let $f_S(x) = Pr[S = x]$. Using Panjer's recursion compute $f_S(x)$, $x = 0, 1, 2, 3, 4, 5, 6$. [20]

4. The Exponential Premium Principle sets a premium, P, based on the utility function $u(x) = (1 - e^{-\alpha x})/\alpha$, where α is a positive constant.

   (a) Show that the premium for a risk S is given by
   $$P = \alpha^{-1} \ln M_S(\alpha)\,,$$
   where $M_S(.)$ is the moment generating function of S, and it is assumed to exist. [15]

   (b) Suppose now that S is a compound Poisson distribution with Poisson parameter λ. Show that α corresponds to the adjustment coefficient as defined in ruin theory. [5]

   (c) For a non-negative S, P has a positive loading, i.e., $P > E[S]$. Show that P approaches the Pure Premium as α approaches 0^+. [5]

5. Let X be a risk whose distribution can be approximated by a $Normal(\mu, \sigma^2)$. Let $\Phi(.)$ and $\phi(.)$ be the distribution and the density functions of the standard Normal, respectively.

   Show that the Conditional Tail Expectation, CTE, for a given probability level p can be translated into the standard deviation principle. Deduce the loading coefficient. [20]

   [Remark: You may use the property $x\phi(x) = -\Phi^{''}(x) = -\phi^{'}(x)$.]

6. Consider an insurer running a portfolio with policy limits and premia as detailed in the next table (unit €10^3). Assume that the insurer only has capacity to retain an amount of 100,000 per risk. A surplus treaty has been agreed with a reinsurer. Historical data, by policy limit, allowed to estimate expected aggregate losses by risk size as shown in the table. Figures do not include commissions or other expenses.

		Gross of reinsurance		
Limit	% ceded	Losses	Premium	Loss Ratio
100		3,500	7,000	
250		1,750	3,000	
400		2,500	3,250	
1,000		750	800	
Total				

(a) Compute the percentage ceded and Loss Ratio of reinsurance for each risk size. [10]

(b) Calculate the Loss Ratio for the retained total aggregate claims and the total ceded claims. [10]

7. Consider the classical Cramér-Lundberg surplus process where the aggregate claim amount per unit time is compound Poisson with $\lambda = 4$,

$$f_X(x) = e^{-2x} + \frac{3}{2}e^{-3x}\,, \quad x > 0\,,$$

and $c = 3$. Premium is calculated according to the expected values principle.

(a) Calculate the premium loading and the moment generating function of aggregate claims per unit time. [15]

(b) Calculate Lundberg's upper bound. [15]

(c) Let $\phi(u) = 1 - \psi(u)$ be the ultimate survival probability with initial surplus u. The survival probability can be expressed as the 'renewal function'

$$\phi(u) = \phi(0) + \int_0^u g(0,x)\phi(u-x)dx, \quad u > 0$$

where $g(u,x)$, $x > 0$, is the (defective) density of the severity of ruin, for a given initial surplus u, $u \geq 0$. Give a careful interpretation for the formula. [10]

Solutions

1.

(a) Yes, it corresponds to a Negative binomial ditr. with $r = 11/3$, $\beta = 1$.

(b) $f(0) = 0; \sum_{x=1}^{\infty} f(x) = 1$. This is a geometric distribution translated to 1. $p_k = 0.9p_{k-1} \to a = 09; b = 0, k = 2, 3, \ldots$ $(a, b, 1)$ class.

(c) $E[z^N] = E[E[z^N|\Lambda]] = E[\exp\{\Lambda(z-1)\}] = M_\Lambda(z-1) = (1-\beta(z-1))^{-\alpha}$.

2.

(a) Let S be the aggregate loss with a deductible. We can express $S = \sum_{i=1}^{N^L} Y_i^L = \sum_{i=1}^{N^P} Y_i^P$, with $N^P = \sum_{i=1}^{N^L} I_i$, $I_i \frown Binomial(1;p)$, $p = 1 - F_X(d)$, since a zero loss do not originate a payment. $N^P \frown Poisson(\lambda^\star = \lambda p)$, so $S \frown CPoisson(\lambda, F_{Y^L})$, or $S \frown CPoisson(\lambda^\star, F_{Y^P})$ with $F_{Y^L}(x) = F_X(x+d)$, $x \geq 0$, and $F_{Y^P}(x) = (F_X(x+d) - F_X(d))/p$, $x > 0$.

(b) $E[Y^L] = \int_{2.5}^{\infty} \exp\{-0.1x\}dx = 10\exp\{-0.25\}$. $E[S^L] = (0.1)(10\exp\{-0.25\}) = \exp\{-0.25\} \simeq 0.77880$.

3. (a) Compute, e.g.

$$\begin{aligned} M_S(r) &= M_{S_1}(r)M_{S_2}(r) = \exp\{(3/2)(M_X(r)-1)\} \\ M_X(r) &= \frac{2}{3}M_{X_1}(r) + \frac{1}{3}M_{X_2}(r) \\ &\Rightarrow f_X(x) = \frac{2}{3}f_1(x) + \frac{1}{3}f_2(x) \end{aligned}$$

(b) Use $f_X^{*k}(x) = \sum_{y=1}^{x-1} f_X^{*(k-1)}(x-y)f_X(y)$ to get,

x	$f^{*0}(x)$	$f^{*1}(x)$	$f^{*2}(x)$	$f^{*3}(x)$
0	1			
1		2/3		
2		1/3	$4/3^2$	
3			$4/3^2$	$8/3^3$
4			$1/3^2$	$12/3^3$
5				$6/3^3$
6				$1/3^3$

(c) We have $f_S(0) = \exp\{-3/2\}$, $f_S(x) = (3/2x)\sum_{k=1}^{x\wedge 2} kf_x(x)f_S(x-k)$, so

x	0	1	2	3	4	5	6
$f_S(x)$	$0.223,130$	$0.223,130$	$0.223,130$	$0.148,753$	$0.092,971$	$0.048,345$	$0.023,553$

4. (a) Develop

$$u(x) = E[u(x+P-S)] \Leftrightarrow \alpha P = \ln M_S(\alpha).$$

(b) If $S \frown CPoisson(\lambda, F_X)$, from above $\alpha P = \lambda(M_X(\alpha)-1)$ or $\alpha P + \lambda = \lambda M_X(\alpha)$, which is the equation that defines the adjustment coefficient.

(c) Let $\phi(.)$ be the cumulant generating function of S. $\lim_{\alpha\downarrow 0} P(\alpha) = 0/0$. Apply l'Hôpital's rule to get $\lim_{\alpha\downarrow 0} P(\alpha) = \lim_{\alpha\downarrow 0}\phi'(\alpha) = E[S]$.

5. Let $Z = (X-\mu)/\sigma)$, $\pi_p : F_X(\pi_p) = \Phi((\pi-\mu)/\sigma) = p$, and $\pi_p^* = (\pi_p - \mu)/\sigma$. Now,

$$CTE = E[X|X > \pi] = \mu + \sigma E[Z|Z > \pi^*],$$

and the loading coefficient

$$E[Z|Z > \pi^*] = \frac{\int_{\pi^*}^{\infty} x\phi(x)dx}{1-p} = \frac{\int_{\pi^*}^{\infty} -\phi'(x)dx}{(1-p)} = \frac{\phi(\pi^*)}{1-p} = \frac{\phi[\Phi^{-1}(p)]}{1-p} > 0.$$

6. (a)

		Gross of reinsurance		
Limit	% ceded	Losses	Premium	Loss Ratio %
100	0	3,500	7,000	50
250	60	1,750	3,000	58
400	75	2,500	3,250	77
1,000	90	750	800	94
Total		8,500	14, 050	60

(b)

		Net of reinsurance	
Limit	% ceded	Losses	Premium
100	0	3,500	7,000
250	60	700	1,200
400	75	625	812.5
1,000	90	75	80
Total		4,900	9,092.5

Loss ratio for the retained business: 54%; Loss ratio for the ceded business: 73%

7.

$$f_X(x) = \frac{1}{2}\left(2e^{-2x}\right) + \frac{1}{2}\left(3e^{-3x}\right)\ , \quad x > 0\,,$$

(a)

$$\begin{aligned} M_X(x) &= \frac{1}{2}\frac{2}{2-r} + \frac{1}{2}\frac{3}{3-r} = \frac{1}{2}\frac{12-5r}{(2-r)(3-r)} \\ M_S(r) &= \exp\{4(M_X(r)-1)\} \\ \mu_X &= \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3} = \frac{5}{12}\,. \end{aligned}$$

Then the loading comes from $4(5/12)(1+\alpha) = 3 \Leftrightarrow \alpha = 4/5$, and loading is $(4/5)(5/3) = 4/3$.

(b) r: $1 + \frac{3}{4}r = \frac{1}{2}\frac{12-5r}{(2-r)(3-r)}$, giving $r = 0 \vee r = 1 \vee r = 8/3$. Then, $R = 1$ and $\psi(u) \leq e^{-u}$.

(c) We can consider the surplus process surviving in two exclusive ways: (i) Surviving its initial level. This has probability $\phi(0)$, or; (ii) down crossing its initial level but by an amount x not greater than u, then re-starting the process from initial surplus $u - x$, surviving from thereon. The process has independent increments.The probability of this event is the integral part. The integral is to consider any value $0 < x < u$.