Master in Actuarial Science
Risk Theory

Exam 2, 05/07/2016
Time allowed: Three hours

Instructions:

1. This paper contains 7 questions and comprises 3 pages including the title page;
2. Enter all requested details on the cover sheet;
3. You have 10 minutes of reading time. You must not start writing your answers until instructed to do so;
4. Number the pages of the paper where you are going to write your answers;
5. Attempt all 7 questions;
6. Begin your answer to each of the 7 questions on a new page;
7. Marks are shown in brackets. Total marks: 200;
8. Show calculations where appropriate;
9. An approved calculator may be used. No mobile phones are permitted;
10. The distributed Formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed Formulary.

1. Consider the (a, b, i), $i = 0, 1$, classes of distributions.

   (a) Do the values $a = 1/2$ and $b = -5/3$ lead to a legitimate distribution of class $(a, b, 0)$? Explain with calculation. [5]

   (b) The recursion $p_k = (a + b/k)\, p_{k-1}$, $k \in \mathbb{N}$, is equivalent to $k\, p_k/p_{k-1} = a\,k + b$, so that the right-hand side is viewed as a linear function of k, $k = 1, 2, \ldots$, where a is the "slope".
   According to "slope" a classify the only members of the $(a, b, 0)$ class. [5]

   (c) Find, and identify, the distribution resulting from mixing a Poisson with an exponential distribution with mean β. Explain/justify every step. [10]

   (d) What sort of resulting distribution you get if you mix a degenerate distribution at the origin with a member of the $(a, b, 0)$ class? Explain shortly. [5]

2. Suppose that in a certain portfolio individual losses are subject to a deductible and coinsurance α. Number of losses per period of time are Poisson distributed with mean 0.15. Let Y^L be the per loss random variable, so that $Y^L = \max(0, \alpha(X - d))$, and X be the the individual claim amount. Single claim amount distribution is Exponential with mean 10, deductible $d = 2.5$ and $\alpha = 0.8$.

   (a) Show that the moment generating function of Y^L is given by

   $$M_{Y^L}(r) = 1 + e^{-0.25} \frac{8\,r}{1 - 8\,r}\,.$$

   [12.5]

   (b) Show that the aggregate payments, on a per payment basis, is a compound Poisson distribution with parameter $0.15 \exp(-0.25)$ and individual payments exponentially distributed with mean 8. [12.5]

3. Consider two independent compound Poisson discrete random variables S_1 and S_2, with Poisson parameters λ_1 and λ_2, and single claim probability functions f_1 and f_2, respectively. Let $\lambda_1 = \lambda_2 = 1/2$, $f_1(1) = f_1(2) = 1/2$, $f_2(1) = 1/3$, $f_2(2) = 2/3$ and $S = 2\,S_1 + S_2$.

   (a) Show that S is compound Poisson with parameter $\lambda = 1$ and single claim amount probability function associated to S, say f_X, is given by the weighted average

   $$f(x) = \frac{1}{2} g(x) + \frac{1}{2} f_2(x) \quad \text{and} \quad g(2) = g(4) = \frac{1}{2}\,.$$

   [15]

   (b) Consider the convolutions $f_X^{*k}(x) = Pr(X_1 + \cdots + X_k = x)$ for $k = 0, 1, 2, 3$ and $x = 0, 1, 2, \ldots, 12$. Fill in the blank cells of the table below. [15]

x	$f_X^{*0}(x)$	$f_X^{*1}(x)$	$f_X^{*2}(x)$	$f_X^{*3}(x)$
0			0	0
1			0	0
2			0.027778	0
3			0.194444	
4			0.340278	
5		0	0.083333	
6		0	0.291667	
7		0	0	
8		0	0.062500	
9		0	0	
10		0	0	
11		0	0	
12		0	0	

   (c) Let $f_S(x) = Pr[S = x]$. Using Panjer's recursion compute $f_S(x)$, $x = 0, 1, 2, 3, 4, 5, 6$. [20]

4. Economic theory in insurance explains why insureds will be willing to pay a premium that may be larger than the pure or net premium. This is based on the expected utility principle. A similar procedure may be used by insurers when charging a premium.

Let P^+ be the maximum premium an individual is accepting to pay for a given risk X, and P^- be the minimum premium an insurer is willing to accept for the same risk. P^+ and P^- can be calculated using the equilibrium equations, with utility functions $u_i(x)$ and wealth w_i, $i = 1, 2$, respectively.

$$\begin{aligned} E[u_1(w_1 - X)] &= u_1(w_1 - P^+)\,, \\ u_2(w_2) &= E[u_2(w_2 + P^- - X)]\,. \end{aligned}$$

(a) Explain/interpret the equations above, assigning them appropriate and clearly to the insurer and the insured. [5]

(b) Suppose that both the insurer and the insured use the same utility function $u(x) = (1 - \exp(-\alpha x)\,\alpha^{-1}$. Show that P^+ and P^- give the same solution and deduce the solution. [15]

(c) Consider the previous $u(x)$. Consider a risk aversion of 0.005 and that X is exponentially distributed with mean 100. Calculate the loading on the resulting net premium. [5]

(d) Consider the previous question. Calculate the loading coefficient if the premium is considered to be calculated according to the variance principle, for the same premium value as above. [5]

5. Let $(X_1, X_2, \ldots, X_n)$ be a set of n correlated risks, each translated to have zero mean, and Normally distributed, $X_i \frown N(0, \sigma_i^2)$, $i = 1, \ldots, n$. Let $Y = \sum_{i=1}^n w_i X_i$ be a linear combination of the n r.v.'s X_i, $i = 1, \ldots, n$. Show that

$$VaR_p(Y) = \sqrt{\sum_{i=1}^n \sum_{j=1}^n \rho_{ij} w_i w_j \, VaR_p(X_i) VaR_p(X_j)}\,,$$

where ρ_{ij} is the linear correlation coefficient of X_i and X_j ($\rho_{ij} = 1$, if $i = j$). [20]

[Remark: $VaR_p(X_i) = \sigma_i \Phi^{-1}(p)$, where $\Phi(\dot{})$ is the distribution function of the standard Normal.]

6. Consider an insurer running a portfolio with policy limits and premia as detailed in the next table (unit €10^3). Assume that the insurer only has capacity to retain an amount of €120,000 per risk. A surplus treaty has been agreed with a reinsurer. Historical data, by policy limit, allowed to estimate expected aggregate losses by risk size as shown in the table. Figures do not include commissions or other expenses.

		Gross of reinsurance		
Limit	% ceded	Losses	Premium	Loss Ratio
120		3,400	7,000	
300		1,750	2,750	
500		2,500	3,250	
1,200		1,000	900	
Total				

(a) Compute the percentage ceded and Loss Ratio of reinsurance for each risk size. [10]

(b) Calculate the Loss Ratio for the retained total aggregate claims and the total ceded claims. [10]

7. Consider the classical Cramér-Lundberg surplus process $\{U(t), t \geq 0\}$ run by equation

$$U(t) = u + ct - S(t)\,, \quad t \geq 0\,,$$

where $u = U(0)$, c is the premium income per unit time and $S(t)$ the aggregate claims up to time t. $\{S(t)\}$ is a compound Poisson process with parameter $\lambda = 4$, and severity distribution

$$f_X(x) = e^{-2x} + \frac{3}{2} e^{-3x}\,, \quad x > 0\,.$$

Premium is calculated according to the expected value principle, with loading coefficient $\theta = 4/5$.

(a) The ultimate ruin probability is of the form

$$\psi(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u}\,, \quad u \geq 0\,,$$

where C_1 and C_2 are constants, that can be calculated.
We can show that $r_1 = 1$ and $r_2 = 8/3$. Explain with appropriate calculation. [15]

(b) Show that $C_1 + C_2 = 5/9$. [5]

(c) It is possible to prove that $\psi(u) \sim Ce^{Ru}$ as $u \to \infty$, where R is the adjustment coefficient and C is a constant (Cramér's asymptotic formula). Show that for this case $C = C_1$. [10]

Solutions

1.

(a) No, we have $p_2/p_1 = a + b = -7/6 < 0$.

(b) If $a < 0$ we get a binomial, if $a = 0$ a Poisson and if $a > 0$ we have a negative binomial (with $b = 0$, a geometric).

(c) Use for instance generating functions: $E[z^N] = E[E[z^N|\Lambda]] = E[\exp\{\Lambda(z-1)\}] = M_\Lambda(z-1) = 1-\beta(z-1)$, where $\Lambda \frown Exp(\beta^{-1})$. $1-\beta(z-1)$ is the m.g.f. of a geometric distribution with mean β.

(d) We get a member of the $(a, b, 1)$ class or "zero modified distr.".

2.

(a)

$$\begin{aligned} M_{Y^L}(r) &= e^0 F_X(2.5) + \int_{2.5}^{+\infty} e^{0.8(x-2.5)r} \left(0.1\, e^{-0.1\, x}\right) dx \\ &= 1 + e^{-0.25} \frac{8\, r}{1-8\, r} \end{aligned}$$

(b) Let S be the aggregate payments.

$$\begin{aligned} M_S(r) &= M_{N^L}\left(M_{Y^L}(r)\right) = e^{\lambda\left(M_{Y^L}-1\right)} \\ &= e^{0.15e^{-0.25}(1/(1-8r)-1)}\,. \end{aligned}$$

$(1-8r)^{-1}$ is the m.g.f. of the individual payments.

3. (a) Compute, e.g.

$$M_S(r) = M_{S_1}(2r)M_{S_2}(r) = \exp\{(1/2)M_1(2r) + (1/2)M_2(r) - 1)\}\,.$$

Now,

$$M_1(2r) = e^{2r}/2 + e^{4r}/2 \Rightarrow g(x) = 1/2\,; x = 2, 4$$

and $f(x) = g(x)/2 + f_2(x)/2$.

(b) Use $f_X^{*k}(x) = \sum_{y=1}^{x-1} f_X^{*(k-1)}(x-y) f_X(y)$,

x	$f_X^{*0}(x)$	$f_X^{*1}(x)$	$f_X^{*2}(x)$	$f_X^{*3}(x)$
0	1	0	0	0
1		1/6	0	0
2		7/12	0.027778	0
3		0	0.194444	0.00462963
4		1/4	0.340278	0.048611111
5		0	0.083333	0.170138889
6		0	0.291667	0.219328704
7		0	0	0.145833333
8		0	0.062500	0.255208333
9		0	0	0.03125
10		0	0	0.109375
11		0	0	0
12		0	0	0.015625

(c) We have $f_S(0) = \exp\{-1\}$, $f_S(x) = (1/x)\sum_{k=1}^{x\wedge 2} k f_x(x) f_S(x-k)$, so

x	0	1	2	3	4	5	6
$f_S(x)$	0.3678794	0,06131324	0.21970578	0.03604991	0.15755280	0.025926054	0.067973063

4. (a) Equation with $u_1(x)$, for the insured, individual. He is willing to buy insurance if on average his utility remains the same, with buying insurance, $u_1(w_1 - P^+)$, or not buying insurance, $u_1(w_1 - X)$.

Equation with $u_2(x)$, for the insurer. He is willing to sell insurance if on average his utility remains the same, with selling insurance, $u_2(w_2 + P^- - X)$, or not selling, where his utility remains unchanged.

(b)

$$\begin{aligned} u(w_1 - P^+) &= E[u(w_1 - X)] \Leftrightarrow E[e^{\alpha P^+}] = E[e^{\alpha X}]\,. \\ u(w_2) &= E[u(w_2 + P^- - X)] \Leftrightarrow E[e^{\alpha P^-}] = E[e^{\alpha X}]\,, \end{aligned}$$

giving $P^+ = P^- = P = \ln E[e^{\alpha X}]/\alpha$. We assume that $M_X(\alpha)$ exist. Note that the initial wealth of both entities does not have to be the same.

(c) $M_X(r) = (1 - 100r)^{-1}$. $P = 100 + 38.6$, loading of 38.6.

(d) $P = \mu_X + \theta V[X] = 100 + 100^2\theta \to 100^2\theta = 38.6 \Leftrightarrow \theta = 3.86‰$

5.

$$Y \frown N(0, \sigma_Y^2) \quad \text{and} \quad \sigma_Y^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j\, \sigma_i \sigma_i \rho_{ij}\,,$$

where $\rho_{ij} = Cov[X_i, X_j]/\sigma_i\sigma_j$, $\rho_{ij} = 1$ and $\sigma_i\sigma_j = \sigma_i^2$ if $i = j$. Then

$$\begin{aligned} VaR_p(Y) &= \sigma_Y \Phi^{-1}(p) = \Phi^{-1}(p)\sqrt{\sum_{i,j=1}^{n} w_i w_j\, \sigma_i \sigma_i \rho_{ij}} \\ &= \sqrt{\sum_{i,j=1}^{n} w_i w_j\, \sigma_i \sigma_i \rho_{ij} [\Phi^{-1}(p)]^2}\,. \end{aligned}$$

Then using $VaR_p(X_i) = \sigma_i \Phi^{-1}(p)$ we get the result.

6. (a)

		Gross of reinsurance		
Limit	% ceded	Losses	Premium	Loss Ratio %
120	0	3,400	7,000	49
300	60	1,750	2,750	64
500	76	2,500	3,250	77
1,200	90	1,000	900	111
Total		8,650	13, 900	62

(b)

		Net of reinsurance	
Limit	% ceded	Losses	Premium
120	0	3,400	7,000
300	60	700	1,100
500	76	600	780
1,200	90	100	90
Total		4,800	8,970

Loss ratio for the retained business: 54%; Loss ratio for the ceded business: 78%

7.

$$f_X(x) = \frac{1}{2}\left(2e^{-2x}\right) + \frac{1}{2}\left(3e^{-3x}\right)\,, \quad x > 0\,,$$

(a)

$$\begin{aligned} M_X(x) &= \frac{1}{2}\frac{2}{2-r} + \frac{1}{2}\frac{3}{3-r} = \frac{1}{2}\frac{12-5r}{(2-r)(3-r)} \\ \mu_X &= \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{3} = \frac{5}{12}\,. \end{aligned}$$

We have a combination of exponentials where the ruin probability will be of the form written, r_1 and r_2 are postive solutions of equation

$$1 + (1+\theta)\mu_X r = M_X(r) \Leftrightarrow 1 + (1 + 4/5)\frac{15}{12}r = \frac{1}{2}\frac{12-5r}{(2-r)(3-r)}\,.$$

giving $r = 1$ and $r = 8/3$.

(b) $\psi(0) = (1+\theta)^{-1} = C_1 + C_2 = 5/9$.

(c) Equate $e^{r_1 u}\psi(u) = C_1 + C_2 e^{-(r_2 - r_1)u}$ and calculate the limit as $u \to \infty$. In the right-hand side the second part tends to zero since $r_2 > r_1$.