Stochastic Calculus - part 1

Master Programme in Mathematical Finance

ISEG

2016

Program

1. Introduction
2. A review of basic concepts in probability and stochastic processes
3. Brownian motion
4. The stochastic integral
5. The Itô formula
6. Stochastic Differential Equations
7. Stochastic differential equations and partial differential equations
8. Grisanov Theorem
9. Application to financial markets and derivatives pricing

Main Bibliography

1. J. Guerra, Cálculo Estocástico, Lecture Notes (Texto de Apoio), 2012.
2. B. Oksendal, Stochastic Differential Equations, Springer, 1998.
3. D. Nualart, Stochastic Calculus (Lecture notes, Kansas University): http://www.math.ku.edu/~nualart/StochasticCalculus.pdf
4. T. Mikosch, Elementary Stochastic Calculus with Finance in view, World Scientific, 1998.

Optional Bibliography

1. Björk, Tomas; Arbitrage Theory in Continuous Time, Oxford University Press, 1998.
2. I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus , 2nd edition, Springer, 1991.
3. P. E. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations , Springer, 1992.
4. F. Klebaner, Introduction to Stochastic Calculus with Applications, 3rd edition, Imperial College Press, 2012.
5. Steven Shreve, Stochastic Calculus for Finance II: Continuous-Time Models, Springer, 2004.

Introduction

- What is stochastic calculus?
- Study of integral (and differential) calculus with respect to stochastic processes.
- We can define integrals of stochastic processes where the "integrating function" is replaced also by a stochastic process
- The most important stochastic process (paradigm): Brownian motion

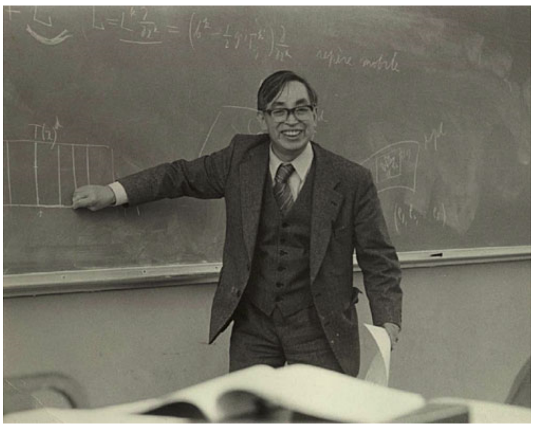
Main topics

1. Construction of stochastic integrals
2. Itô formula
3. Stochastic differential equations
4. Partial differential equations and their relationships with stochastic differential equations
5. Stochastic calculus for Lévy processes, semimartingales and some processes which are not semimartingales (like the fractional Brownian motion)
6. Stochastic partial differential equations
7. Applications to Finance, Physics, Biology, Economics, etc...

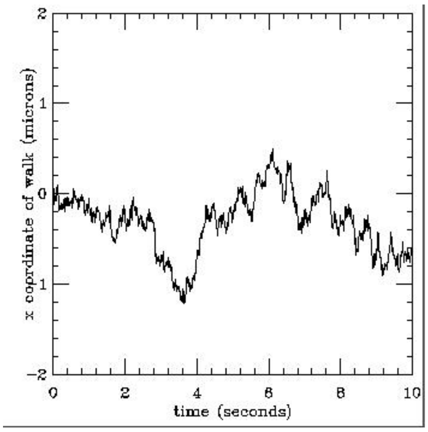
Stochastic Calculus Heroes

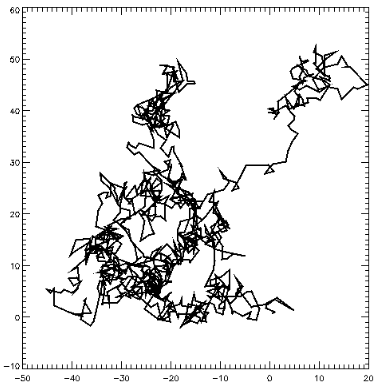
See Robert Jarrow and Philip Protter: "A short History of Stochastic Integration and Mathematical Finance" in A festschrift for Herman Rubin, 75–91, IMS Lecture Notes Monogr. Ser., 45, Inst. Math. Statist., Beachwood, OH, 2004.
Download from http://projecteuclid.org/euclid.lnms/1196285381

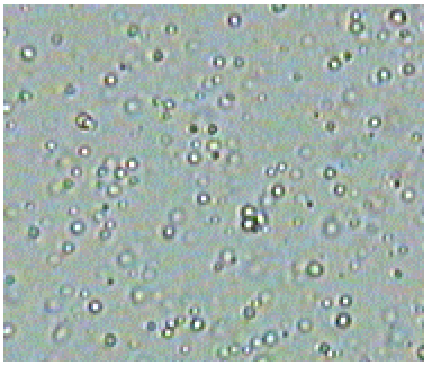
- T. N. Thiele
- Louis Bachelier
- Albert Einstein
- Norbert Wiener
- Kolmogorov
- Vincent Doeblin
- Kiyosi Itô
- Doob
- P. A. Meyer
- Malliavin
- etc...



Brownian motion







Stochastic Processes

Definition

Stochastic Process: is a family of random variables $\{X_t, t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. T: set where the parameter t is defined. If $T = \mathbb{N}$, we have a discrete time process.If $T = [a, b] \subset \mathbb{R}$ or if $T = \mathbb{R}$, we have a continuous time process.

- $\{X_t, t \in T\} = \{X_t(\omega), \omega \in \Omega, \; t \in T\}$
- X_t: state or position of the process at time t.
- The space of states (space where the r.v. have values) is usually $\mathbb{R}$ (continuous state space) or $\mathbb{N}$ (discrete state space).
- For each fixed ω ($\omega \in \Omega$), the map $t \rightarrow X_t(\omega)$ or $X_{\cdot}(\omega)$ is called a trajectory or sample path of the process.

Example

Random walk: Consider a sequence of independent r.v. $\{Z_t, t \in \mathbb{N}\}$. Then

$$X_t = Z_1 + Z_2 + \cdots Z_t = X_{t-1} + Z_t$$

is a stochastic process in discrete time (the random walk).

Example

A Markov processis a process in which the probability of a future state in time t depends only on the state previously observed at time t_k, i.e., if $t_1 < t_2 < \cdots < t_k < t$, then

$$P\left[a < X_t < b | X_{t_1} = x_1, X_{t_2} = x_2, \ldots, X_{t_k} = x_k\right] = P\left[a < X_t < b | X_{t_k} = x_k\right].$$

A Markov process with discrete state space is a Markov chain. If it has continuous state space and is in continuous time, then it is a diffusion.

- Probabilistic characterization of a process X? .

Definition

Let $\{X_t, t \in T\}$ be a stochastic process. The finite dimensional distributions (or fidis) of X are all the distributions of vectors

$$\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right),$$

where $n = 1, 2, 3, \ldots; t_1, t_2, \ldots, t_n \in T$.

- Distribution of the stochastic process $\approx$ fidis.

Example

Gaussian process: when all the fidis are Gaussian. If we know μ (mean) and Σ (covariance function or matrix) we can characterize completely the Gaussian distribution. Therefore, in order to know a Gaussian process, we only need to know μ and Σ.

Example

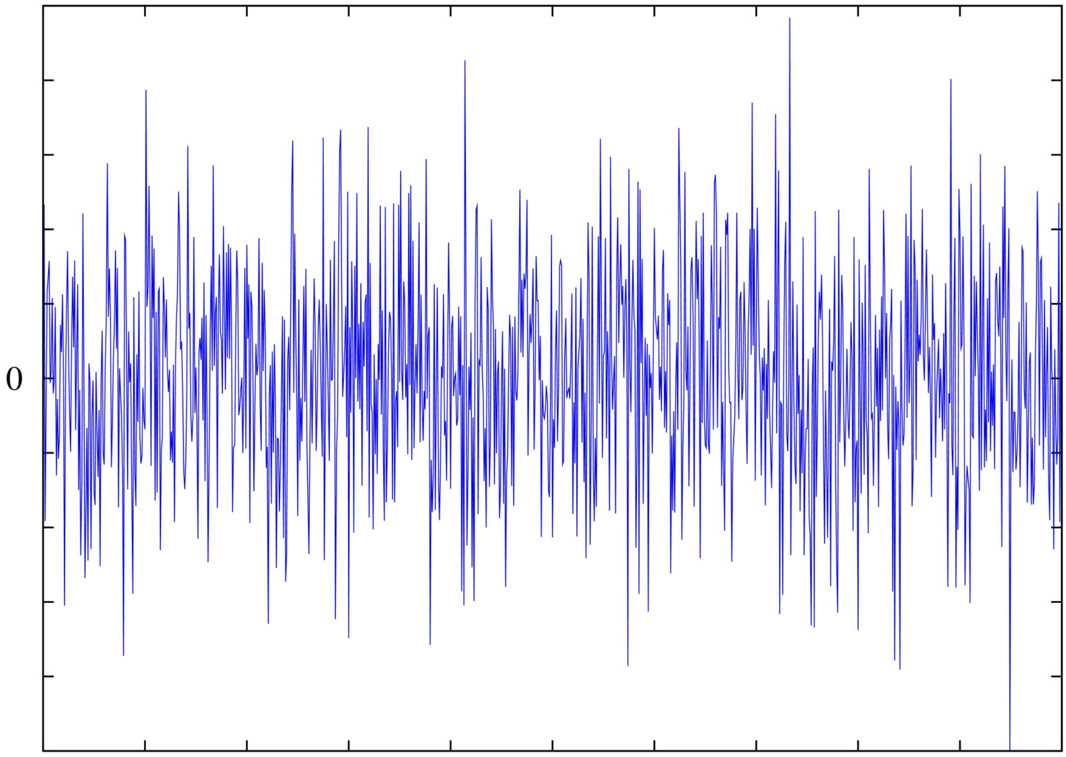
(white noise) Let $\{X_t, t \geq 0\}$ and $X_t \sim N(0, \sigma^2)$, with all the r.v. independent. Then, the process is Gaussian and the fidis are associated to the distribution functions

$$\begin{aligned} F(x_1, x_2, \ldots, x_n) &= P\left(X_{t_1} \leq x_1, X_{t_2} \leq x_2, \ldots, X_{t_n} \leq x_n\right) \\ &= P(X_{t_1} \leq x_1) P\left(X_{t_2} \leq x_2\right) \ldots P\left(X_{t_n} \leq x_n\right) \\ &= \Phi(x_1) \Phi\left(x_2\right) \ldots \Phi\left(x_n\right). \end{aligned}$$

The expected value and the covariance function of X are:

$$\mu_X(t) = E[X_t] = 0,$$

$$c_X(s,t) = E\left[\left(X_t - \mu_X(t)\right)\left(X_s - \mu_X(s)\right)\right] = \begin{cases} \sigma^2 & \text{se } s = t \\ 0 & \text{se } s \neq t \end{cases}.$$



- In general,

$$\mu_X(t) = E[X_t],$$
$$c_X(s,t) = cov(X_t, X_s) = E[(X_t - \mu_X(t))(X_s - \mu_X(s))].$$

Definition

A stochastic process X is said to be strictly (or strongly) stationary if

$$(X_{t_1}, X_{t_2}, \dots, X_{t_n}) \stackrel{d}{=} (X_{t_1+h}, X_{t_2+h}, \dots, X_{t_n+h}),$$

for all the possible choices of n; $t_1, t_2, \dots, t_n \in T$ and h.

Definition

A stochastic process X has stationary increments if

$$X_t - X_s \stackrel{d}{=} X_{t+h} - X_{s+h},$$

for all the possible values of s, t and h.

- Exercise: Show that if X is a Gaussian and stongly stationary process then $\mu_X(t) = \mu_X(0)$, $\forall t \in T$ and $c_X(s,t) = f(|s-t|)$ just depends on the distance $|s-t|$.

Definition

A stochastic process has independent increments if the random variables

$$X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$$

are independent whenever $t_1 < t_2 < \dots < t_n$, $n = 1, 2, \dots$

- All the processes with independent increments are Markov processes.

Example

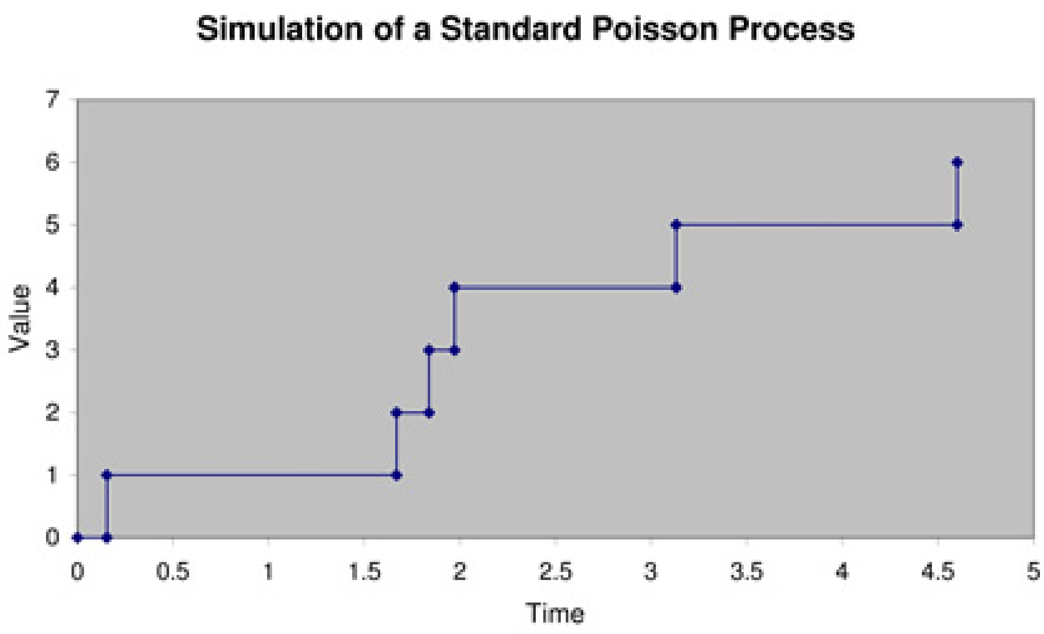
(Poisson Process) A s. p. $\{X_t, t \geq 0\}$ is a Poisson process with intensity λ if

- $X_0 = 0$,
- X has stationary and independent increments,
- $X_t \sim Poi(\lambda t)$.

- If $Y \sim Poi(\lambda)$ then

$$P(Y = k) = e^{-\lambda} \frac{\lambda^k}{k!}.$$

- Exercise: Show that if X is a Poisson process then $X_t - X_s \sim Poi(\lambda(t - s))$ if $t > s$.



Definition

A s. p. $\{X_t, t \in T\}$ is said to be equivalent to the s. p. $\{Y_t, t \in T\}$ if, for each $t \in T$, we have

$$P\{X_t = Y_t\} = 1.$$

In this case, we say that process X is a version of process Y.

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- Two equivalent processes can have very different trajectories or paths.

Example

Let φ be a non-negative r. v. with a continuous distribution and consider the s. p.

$$X_t = 0,$$
$$Y_t = \begin{cases} 0 & \text{se } \varphi \neq t \\ 1 & \text{se } \varphi = t \end{cases}$$

The s. p. are equivalent. However, their trajectories are different.

Definition

Two s. p. $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ are said to be undistinguishable if

$$X_{\cdot}(\omega) = Y_{\cdot}(\omega) \quad \forall \omega \in \Omega \backslash N,$$

where N has zero probability ($P(N) = 0$).

- Two s. p. with continuous trajectories which are equivalent are also undistinguishable.