Stochastic Calculus - part 4

ISEG

2016

Brownian motion

Definition

A s.p. $B = \{B_t; t \geq 0\}$ is a Brownian motion if

1. $B_0 = 0$.
2. B has independent and stationary increments.
3. If $s < t$, $B_t - B_s$ is a r.v. with distribution $N(0, t - s)$.
4. The process B has continuous trajectories.

Brownian motion properties

- The B.m. is a Gaussian process. Indeed, The finite dimensional distributions of B, i.e. the distribution of the vectors $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$ is a multivariate Gaussian distribution.
- $E[B_t] = 0$ (consequence of condition 3).
- Covariance function: $c(s,t) = E[B_s B_t] = \min(s,t)$

Proof.
If $s \le t$:

$$\begin{aligned} E[B_s B_t] &= E\left[B_s(B_t - B_s) + B_s^2\right] \\ &= E[B_s(B_t - B_s)] + E\left[B_s^2\right] \\ &= E[B_s]\,E[B_t - B_s] + s = s. \end{aligned}$$

□

- A s.p. that satisfies properties 1,2 and 3 has a version with continuous trajectories.

Proof.
Since $(B_t - B_s) \sim N(0, t-s)$, it is possible to show that

$$E\left[(B_t - B_s)^{2k}\right] = \frac{(2k)!}{2^k \cdot k!}(t-s)^k.$$

In order to prove this formula, one can use integration by parts and the mathematical induction principle in k (see Oksendal book) With $k = 2$:

$$E\left[(B_t - B_s)^4\right] = 3(t-s)^2.$$

Kolmogorov continuity criterion $\Longrightarrow$exists a version of B with cont. paths.

□

- Exists a s.p. that satisfies conditions 1,2,3 and 4 (by the Kolmogorov extension Theorem (see Oksendal book) and by the Kolmogorov continuity criterion).

- In the definition of Brownian motion, the probability space is arbitrary. However, one can describe the structure of this space, considering the map:

$$\begin{aligned} \Omega &\rightarrow C\left([0,\infty),\mathbb{R}\right) \\ \omega &\rightarrow B_{\cdot}\left(\omega\right) \end{aligned}$$

that associates to each ω a continuous function with values in $\mathbb{R}$ (the trajectory). The probability space can therefore be identified with the space of continuous functions $C\left([0,\infty),\mathbb{R}\right)$ equipped with the Borel σ-algebra $\mathcal{B}_C$ and with the probability induced by the previous map: $P_W = P \circ B^{-1}$ (this probability measure is called the Wiener measure). The canonical probability space for the Brownian motion is then the space $\left(C\left([0,\infty),\mathbb{R}\right),\mathcal{B}_C,P_W\right)$. In this space, the random variables are $X_t\left(\omega\right)=\omega\left(t\right)$.

- As a corollary to the Kolmogorov continuity criterion and the formula $E\left[\left(B_t-B_s\right)^{2k}\right]=\frac{(2k)!}{2^k\cdot k!}\left(t-s\right)^k$, we have that

$$\left|B_t\left(\omega\right)-B_s\left(\omega\right)\right| \leq G_\varepsilon\left(\omega\right)\left|t-s\right|^{\frac{1+\alpha}{p}-\varepsilon} \leq G_\varepsilon\left(\omega\right)\left|t-s\right|^{\frac{1}{2}-\varepsilon},$$

for each $\varepsilon>0$, where $G_\varepsilon\left(\omega\right)$ is a r.v.
- Intuitively: $\left|B_t-B_s\right| \approx \left|t-s\right|^{\frac{1}{2}}$
- Moreover: $E\left[\left(B_{t+\Delta t}-B_t\right)^2\right]=\Delta t$
- Consider the interval $[0,t]$ and partitions of this interval $0=t_0<t_1<\cdots<t_n=t$ with $t_j=\frac{tj}{n}$. Then:

1. The B.m. infinite total variation: $\sum_{k=1}^{n}\left|\Delta B_k\right| \approx n\left(\frac{t}{n}\right)^{1/2} \rightarrow \infty$, when $n\rightarrow\infty$.
2. The B.m. has finite quadratic variation: $\sum_{k=1}^{n}\left|\Delta B_k\right|^2 \approx n\left(\frac{t}{n}\right)=t$.

- The trajectories of the B.m. are not differentiable at any point (a.s.). sketch of the proof:

$$\frac{B_{t+\Delta t} - B_t}{\Delta t} \approx \frac{\sqrt{\Delta t} Z}{\Delta t} = \frac{Z}{\sqrt{\Delta t}},$$

where $Z \sim N(0,1)$.This ratio tends to ∞ when $\Delta t \to 0$ in probability, since $P\left(\left|\frac{Z}{\sqrt{\Delta t}}\right| > K\right) \to 1$ for every K, when $\Delta t \to 0$. Therefore, the derivative does not exist at t.
- Self-similarity: If $B = \{B_t; t \geq 0\}$ is a B.m. then, for any $a > 0$, the process $\left\{a^{-1/2} B_{at}; t \geq 0\right\}$ is also a B.m.
- Exercise: show that $\left\{a^{-1/2} B_{at}; t \geq 0\right\}$ satisfies the definition of Brownian motion.

Processes related to Brownian motion

- Brownian motion with drift:

$$Y_t = \mu t + \sigma B_t,$$

with $\sigma > 0$ and μ real constants. Clearly, it is a Gaussian process with $E[Y_t] = \mu t$ and $c(s,t) = E[(Y_s - E[Y_s])(Y_t - E[Y_t])] = \sigma^2 \min(s,t)$.
- Geometric Brownian motion: (proposed by Samuelson, and later used by Black, Scholes and Merton for modeling asset prices)

$$X_t = e^{\mu t + \sigma B_t}.$$

The distribution of X is lognormal, i.e. $\ln(X_t)$ has normal distribution.
- Brownian bridge:

$$Z_t = B_t - tB_1, \quad t \in [0,1].$$

Note that $Z_1 = Z_0 = 0$. This is a Gaussian process with $E[Z_t] = 0$ and $c(s,t) = E[Z_s Z_t] = \min(s,t) - st$.

Martingales and the Brownian motion

- Consider a Brownian motion $B = \{B_t; t \geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$.
- The filtration generated by B is $\{\mathcal{F}_t^B, t \geq 0\}$ with

$$\mathcal{F}_t^B = \sigma \{B_s, s \leq t\} .$$

- Consider that $\mathcal{F}_t^B$ also contains the events of zero probability (consider that $N \in \mathcal{F}_0$ if $P(N) = 0$).

- Some consequences of the inclusion of events of probability 0 in the filtration:
  1. Any version of an adapted process is also an adapted process.
  2. The filtration is right-continuous, i.e.

$$\bigcap_{s>t} \mathcal{F}_s^B = \mathcal{F}_t^B .$$

  (the intersection of all “future information” is the “present information”).
- Example: If B is a B.m. then the process $X_t = \sup_{0 \leq s \leq t} B_s$ is adapted to $\{\mathcal{F}_t^B, t \geq 0\}$ but the process $Y_t = B_{t+\varepsilon}$, with $\varepsilon > 0$, is not.

Proposition

If $B = \{B_t; t \geq 0\}$ is a B.m. and $\{\mathcal{F}_t^B, t \geq 0\}$ is the filtration generated by B, then the following processes are $\{\mathcal{F}_t^B, t \geq 0\}$-martingales:

1. B_t.
2. $B_t^2 - t$.
3. $\exp\left(aB_t - \frac{a^2 t}{2}\right)$.*(Homework: show that it is a martingale)*

Proof.

1. Clearly B_t is $\mathcal{F}_t^B$-measurable and integrable. Moreover, $B_t - B_s$ is independent of $\mathcal{F}_s^B$ (by the independence of the increments of B)

$$E\left[B_t - B_s | \mathcal{F}_s^B\right] = E\left[B_t - B_s\right] = 0.$$

2. Clearly, $B_t^2 - t$ is $\mathcal{F}_t^B$-measurable and integrable. By the properties of B and of the conditional expectation:

$$\begin{aligned} E\left[B_t^2 - t | \mathcal{F}_s^B\right] &= E\left[(B_t - B_s + B_s)^2 | \mathcal{F}_s^B\right] - t \\ &= E\left[(B_t - B_s)^2\right] + 2B_s E\left[B_t - B_s | \mathcal{F}_s^B\right] + B_s^2 - t \\ &= t - s + B_s^2 - t = B_s^2 - s. \end{aligned}$$

□

Quadratic variation of Brownian motion

Proposition

Consider the interval $[0, t]$ *and partitions of this interval* $0 = t_0 < t_1 < \cdots < t_n = t$, *with* $t_j = \frac{tj}{n}$. *Then*

$$E\left[\left(\sum_{k=1}^{n}(\Delta B_k)^2 - t\right)^2\right] \to 0, \text{ when } n \to \infty.$$

and therefore the quadratic variation of the B.m. is finite (it is t *in the interval* $[0, t]$*).*

Proof.

By the independence of the increments and the fact that $E\left[(\Delta B_k)^2\right] = \frac{t}{n}$, we have

$$E\left[\left(\sum_{k=1}^{n}(\Delta B_k)^2 - t\right)^2\right] = E\left[\left(\sum_{k=1}^{n}\left[(\Delta B_k)^2 - \frac{t}{n}\right]\right)^2\right]$$
$$= \sum_{k=1}^{n} E\left[(\Delta B_k)^2 - \frac{t}{n}\right]^2.$$

Using $E\left[(B_t - B_s)^{2j}\right] = \frac{(2j)!}{2^j \cdot j!}(t-s)^j$, we have

$$E\left[\left(\sum_{k=1}^{n}(\Delta B_k)^2 - t\right)^2\right] = \sum_{j=1}^{n}\left[3\left(\frac{t}{n}\right)^2 - 2\left(\frac{t}{n}\right)^2 + \left(\frac{t}{n}\right)^2\right]$$
$$= 2\sum_{i=1}^{n}\left(\frac{t}{n}\right)^2 = 2t\left|\frac{t}{n}\right| \underset{n\to\infty}{\longrightarrow} 0.$$

Total variation of Brownian motion

Proposition

Consider the interval $[0, t]$ *and partitions* π *of this interval:*
$0 = t_0 < t_1 < \cdots < t_n = t$*. Then*

$$V := \sup_{\pi} \sum_{k=1}^{n} |\Delta B_k| = +\infty,$$

a.s. and therefore the total variation of the B.m. is infinite.

Proof.

By the continuity of the B.m. trajectories, and assuming finite V,

$$\sum_{k=1}^{n} (\Delta B_k)^2 \leq \sup_k |\Delta B_k| \sum_{k=1}^{n} |\Delta B_k| \leq V \sup_k |\Delta B_k| \underset{|\pi|\to 0}{\longrightarrow} 0,$$

but $\sum_{k=1}^{n} (\Delta B_k)^2$ converges in quadratic mean to t. Hence, $V = \infty$. □