Stochastic Calculus - part 7

ISEG

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Itô formula

- The Itô fórmula is a stochastic version of the "chain rule"
- Example:

$$\int_0^t B_s dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t$$
$$B_t^2 = 2\int_0^t B_s dB_s + t$$
$$d\left(B_t^2\right) = 2B_t dB_t + dt$$

- $\approx$ Taylor expansion of B_t^2 as a function of B_t and using $\left(dB_t\right)^2 = dt$:

$$\begin{aligned} B_{t+dt}^2 &= B_t^2 + 2B_t dB_t + \frac{1}{2}(2)\left(dB_t\right)^2 \\ &= B_t^2 + 2B_t dB_t + dt. \end{aligned}$$

Itô process

- If f is a function of class C^2, the Itô formula will show that

$$f\left(B_t\right) = \text{indefinite stoch. integral} + \text{process with differentiable paths}$$
$$:= \text{Itô process}$$

- We denote by $L^1_{a,T}$ the space of processes v such that:

1) v is measurable,
2) $P\left[\int_0^T |v_t|\, dt < \infty\right] = 1.$

Definition

A continuous and adapted process $X = \{X_t, 0 \leq t \leq T\}$ is called a Itô process if

$$X_t = X_0 + \int_0^t u_s dB_s + \int_0^t v_s ds, \tag{1}$$

where $u \in L_{a,T}$ and $v \in L^1_{a,T}$.

The One Dimensional Itô formula

One dimensional Itô formula

Theorem

(Itô formula - 1D): Let $X = \{X_t, 0 \leq t \leq T\}$ be a Itô process as in (1). Let $f(t,x)$ be a $C^{1,2}$ function. Then $Y_t = f(t, X_t)$ is a Itô process and

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, u_s dB_s$$
$$+ \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, v_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, u_s^2 ds.$$

- In differential form, the Itô formula reads

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\, dt + \frac{\partial f}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\,(dX_t)^2,$$

where $(dX_t)^2$ is calculated using the product table

$\times$	dt	dB_t
dt	0	0
dB_t	0	dt

- The Itô formula for $f(t, x)$ and $X_t = B_t$, i.e. for $Y_t = f(t, B_t)$:

$$f(t, B_t) = f(0, 0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds.$$

$$df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)\, dt + \frac{\partial f}{\partial x}(t, B_t)\, dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)\, dt.$$

- The Itô formula for $f(x)$ and $X_t = B_t$, i.e. for $Y_t = f(B_t)$:

$$df(B_t) = \frac{\partial f}{\partial x}(B_t)\,dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(B_t)\,dt.$$

The multidimensional Itô formula

- Assume that $B_t := \left(B_t^1, B_t^2, \ldots, B_t^m\right)$ is a Brownian motion of dimension m, i.e., the components B_t^k, $k = 1, ..., m$ are independent one dimensional Brownian motions.
- Consider a Itô process of dimension n, defined by

$$\begin{aligned}
X_t^1 &= X_0^1 + \int_0^t u_s^{11} dB_s^1 + \cdots + \int_0^t u_s^{1m} dB_s^m + \int_0^t v_s^1 ds, \\
X_t^2 &= X_0^2 + \int_0^t u_s^{21} dB_s^1 + \cdots + \int_0^t u_s^{2m} dB_s^m + \int_0^t v_s^2 ds, \\
&\vdots \\
X_t^n &= X_0^n + \int_0^t u_s^{n1} dB_s^1 + \cdots + \int_0^t u_s^{nm} dB_s^m + \int_0^t v_s^n ds.
\end{aligned}$$

The multidimensional Itô formula

- In differential form:

$$dX_t^i = \sum_{j=1}^{m} u_t^{ij} dB_t^j + v_t^i dt,$$

$i = 1, 2, \ldots, n.$

- Or in vector-matrix (compact) form

$$dX_t = u_t dB_t + v_t dt,$$

where v_t is a n-dimensional vector, u_t is a $n \times m$ matrix of processes.

- We assume that the components of u belong to $L_{a,T}$ and the components of v belong to $L_{a,T}^1$

The multidimensional Itô formula

The multidimensional Itô formula

- If $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^p$ is of class $C^{1,2}$ then $Y_t = f(t, X_t)$ is a Itô process and we have the Itô formula

$$dY_t^k = \frac{\partial f_k}{\partial t}(t, X_t)\, dt + \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i}(t, X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(t, X_t)\, dX_t^i dX_t^j.$$

The multidimensional Itô formula

- The product of differentials $dX_t^i dX_t^j$ is calculated using the rules:

$$dB_t^i dB_t^j = \begin{cases} 0 & \text{se } i \neq j \\ dt & \text{se } i = j \end{cases},$$
$$dB_t^i dt = 0,$$
$$(dt)^2 = 0.$$

The multidimensional Itô formula

- If B_t is a n-dimensional Brownian motion and $f : \mathbb{R}^n \to \mathbb{R}$ is of class C^2 with $Y_t = f(B_t)$ then the Itô formula reads

$$f(B_t) = f(B_0) + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(B_t)\, dB_s^i + \frac{1}{2} \int_0^t \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(B_t) \right) ds$$

Integration by parts formula

- We have that

$$dX_t^i dX_t^j = \sum_{k=1}^{m} u_t^{ik} u_t^{jk} dt = \left[u_t \left(u_t \right)^T \right]_{ij} dt.$$

- Integration by parts formula: If X_t^1 and X_t^2 are Itô processes and $Y_t = X_t^1 X_t^2$, then by Itô formula applied to $f(x) = f(x_1, x_2) = x_1 x_2$, we have

$$d\left(X_t^1 X_t^2\right) = X_t^2 dX_t^1 + X_t^1 dX_t^2 + dX_t^1 dX_t^2.$$

  That is:

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_s^2 dX_s^1 + \int_0^t X_s^1 dX_s^2 + \int_0^t dX_s^1 dX_s^2.$$

Example

Example

Consider the process (also called a Bessel process)

$$Y_t = \left(B_t^1\right)^2 + \left(B_t^2\right)^2 + \cdots + \left(B_t^n\right)^2.$$

Let us represent Y in terms of stochastic integrals with respect to the n-dimensional Brownian mot.
By Itô formula applied to $f(x) = f(x_1, x_2, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ we obtain

$$dY_t = 2B_t^1 dB_t^1 + \cdots + 2B_t^n dB_t^n \\ + ndt.$$

That is:

$$Y_t = 2\int_0^t B_s^1 dB_s^1 + \cdots + 2\int_0^t B_s^n dB_s^n + nt.$$

Exercise:

- Let $B_t := \left(B_t^1, B_t^2\right)$ be a two-dimensional Brownian motion. Represent the process

$$Y_t = \left(B_t^1 t, \left(B_t^2\right)^2 - B_t^1 B_t^2\right)$$

as an Itô process.

- Answer: By the Itô multidimensional formula, with $f(t,x) = f(t, x_1, x_2) = \left(x_1 t, x_2^2 - x_1 x_2\right)$, we obtain:

$$\begin{aligned} dY_t^1 &= B_t^1 dt + t dB_t^1, \\ dY_t^2 &= -B_t^2 dB_t^1 + \left(2B_t^2 - B_t^1\right) dB_t^2 + dt. \end{aligned}$$

Or:

$$\begin{aligned} Y_t^1 &= \int_0^t B_s^1 ds + \int_0^t s dB_s^1, \\ Y_t^1 &= -\int_0^t B_s^2 dB_s^1 + \int_0^t \left(2B_s^2 - B_s^1\right) dB_s^2 + t. \end{aligned}$$

Sketch of the proof of the one-dimensional Itô formula

- The process

$$Y_t = f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, u_s dB_s + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, v_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, u_s^2 ds.$$

is an Itô process.

- We assume that f and its derivatives are bounded (the general case can be proves, approximating f and its derivatives by bounded functions).
- As we know, the stochastic integral can be approximated by a sequence of stochastic integrals of simple processes and therefore we can assume that u and v are simple processes.

- Splitting the interval $[0, t]$ into n equal sized sub-intervals, we can write:

$$f(t, X_t) = f(0, X_0) + \sum_{k=0}^{n-1} \left(f\left(t_{k+1}, X_{t_{k+1}}\right) - f\left(t_k, X_{t_k}\right) \right).$$

- By the Taylor formula of f:

$$f\left(t_{k+1}, X_{t_{k+1}}\right) - f\left(t_k, X_{t_k}\right) = \frac{\partial f}{\partial t}\left(t_k, X_{t_k}\right) \Delta t + \frac{\partial f}{\partial x}\left(t_k, X_{t_k}\right) \Delta X_k$$
$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\left(t_k, X_{t_k}\right) \left(\Delta X_k\right)^2 + Q_k,$$

where Q_k is the remainder of the Taylor formula.

Proof of the Itô formula

- We also have that

$$\Delta X_k = X_{t_{k+1}} - X_{t_k} = \int_{t_k}^{t_{k+1}} v_s ds + \int_{t_k}^{t_{k+1}} u_s dB_s$$
$$= v(t_k) \Delta t + u(t_k) \Delta B_k + S_k,$$

where S_k is the error term or remainder.
- From here, we get:

$$(\Delta X_k)^2 = (v(t_k))^2 (\Delta t)^2 + (u(t_k))^2 (\Delta B_k)^2$$
$$+ 2v(t_k) u(t_k) \Delta t \Delta B_k + P_k,$$

where P_k is the remainder term.

- Replacing all these terms, we get

$$f(t, X_t) - f(0, X_0) = I_1 + I_2 + I_3 + \frac{1}{2} I_4 + \frac{1}{2} K_1 + K_2 + R,$$

where

$$\begin{aligned} I_1 &= \sum_k \frac{\partial f}{\partial t}(t_k, X_{t_k}) \Delta t, \\ I_2 &= \sum_k \frac{\partial f}{\partial t}(t_k, X_{t_k}) v(t_k) \Delta t, \\ I_3 &= \sum_k \frac{\partial f}{\partial x}(t_k, X_{t_k}) u(t_k) \Delta B_k, \\ I_4 &= \sum_k \frac{\partial^2 f}{\partial x^2}(t_k, X_{t_k}) (u(t_k))^2 (\Delta B_k)^2 . \end{aligned}$$

- $$\begin{aligned} K_1 &= \sum_k \frac{\partial^2 f}{\partial x^2}(t_k, X_{t_k}) (v(t_k))^2 (\Delta t)^2 , \\ K_2 &= \sum_k \frac{\partial^2 f}{\partial x^2}(t_k, X_{t_k}) v(t_k) u(t_k) \Delta t \Delta B_k, \\ R &= \sum_k (Q_k + S_k + P_k) . \end{aligned}$$

- When $n \to \infty$, one can show that

$$I_1 \to \int_0^t \frac{\partial f}{\partial t}(s, X_s)\, ds,$$
$$I_2 \to \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, v_s ds,$$
$$I_3 \to \int_0^t \frac{\partial f}{\partial x}(s, X_s)\, u_s dB_s.$$

- By the quadratic variation of the Brownian motion, we have

$$\sum_k (\Delta B_k)^2 \to t,$$

and therefore

$$I_4 \to \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\, u_s^2 ds.$$

Proof of the Itô formula

- On the other hand, we also have that

$$K_1 \to 0,$$
$$K_2 \to 0.$$

- One can also show (it is rather technical and more difficult) that

$$R \to 0.$$

- Conclusion: in the limit $n \to \infty$, we obtain the Itô formula.