Stochastic Calculus - part 8

ISEG

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Itô integral representation Theorem

- Let $u \in L^2_{a,T}$ and consider the process

$$M_t = \mathbb{E}[M_0] + \int_0^t u_s dB_s. \quad (1)$$

- We know that M_t is a $\{\mathcal{F}_t\}$-martingale.
- Let us show that any square-integrable martingale has a representation of type (1).

Itô integral representation Theorem

Theorem

(Itô represent. Theorem): Let $F \in L^2(\Omega, \mathcal{F}_T, P)$. Then exists one and only one process $u \in L^2_{a,T}$ such that

$$F = \mathbb{E}[F] + \int_0^t u_s dB_s. \tag{2}$$

Proof: The proof has 3 main steps

1. Assume that

$$F = \exp\left(\int_0^T h(s)\, dB_s - \frac{1}{2}\int_0^T h(s)^2\, ds\right), \tag{3}$$

with deterministic h such that $\int_0^T h(s)^2\, ds < \infty$.

Proof of the Itô repres. Theorem

(Proof-Cont.)
Apply the Itô formula to $f(x) = e^x$, with
$X_t = \int_0^t h(s)\, dB_s - \frac{1}{2}\int_0^t h(s)^2\, ds$ and $Y_t = f(X_t)$.
Then

$$\begin{aligned} dY_t &= Y_t\left(h(t)\, dB_t - \frac{1}{2}h(t)^2\, dt\right) + \frac{1}{2}Y_t\,(h(t)\, dB_t)^2 \\ &= Y_t h(t)\, dB_t. \end{aligned}$$

That is

$$Y_t = 1 + \int_0^t Y_s h(s)\, dB_s.$$

Conclusion:

$$\begin{aligned} F = Y_T &= 1 + \int_0^T Y_s h(s)\, dB_s \\ &= \mathbb{E}[F] + \int_0^T Y_s h(s)\, dB_s \end{aligned}$$

(Proof - Cont.) Note that

$$\mathbb{E}\left[\int_0^T (Y_s h(s))^2 \, ds\right] < \infty,$$

since $\mathbb{E}\left[Y_t^2\right] = \exp\left(\int_0^t h(u)^2 \, du\right) < \infty$. Therefore

$$\begin{aligned}\mathbb{E}\left[\int_0^T (Y_s h(s))^2 \, ds\right] &\leq \int_0^T \exp\left(\int_0^s h(u)^2 \, du\right) h(s)^2 \, ds \\ &\leq \exp\left(\int_0^T h(u)^2 \, du\right) \int_0^T h(s)^2 \, ds.\end{aligned}$$

Itô Integral Representation Theorem

(Proof - Cont.)

2 The representation (2) is also valid (by linearity) for linear combinations of r.v. of form (3).
In the general case, $F \in L^2(\Omega, \mathcal{F}_T, P)$ can be approximated (in the square-mean sense) by a sequence $\{F_n\}$ of linear combinations of r.v. of form (3) (see Oksendal (lemma 4.3.2)). We have that

$$F_n = \mathbb{E}[F_n] + \int_0^t u_s^{(n)} dB_s.$$

By the Itô isometry,

$$\begin{aligned}\mathbb{E}\left[(F_n - F_m)^2\right] &= (\mathbb{E}[F_n - F_m])^2 + \mathbb{E}\left[\int_0^t \left(u_s^{(n)} - u_s^{(m)}\right)^2 ds\right] \\ &\geq \mathbb{E}\left[\int_0^t \left(u_s^{(n)} - u_s^{(m)}\right)^2 ds\right].\end{aligned}$$

(Proof- Cont.)
$\{F_n\}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}_T, P)$. Therefore

$$\mathbb{E}\left[(F_n - F_m)^2\right] \longrightarrow 0 \text{ when } n, m \to \infty.$$

Therefore:

$$\mathbb{E}\left[\int_0^t \left(u_s^{(n)} - u_s^{(m)}\right)^2 ds\right] \longrightarrow 0 \text{ when } n, m \to \infty.$$

Hence, $\left\{u^{(n)}\right\}$ is a Cauchy sequence in $L^2([0,T] \times \Omega)$. Since this is a complete space, we have that $u^{(n)} \to u$ in $L^2([0,T] \times \Omega)$.
The process u is adapted since $u^{(n)} \in L^2_{a,T}$ and exists a subsequence $\left\{u^{(n)}(t,\omega)\right\}$ that converges to $u(t,\omega)$ for a.a. $(t,\omega) \in [0,T] \times \Omega$.
Therefore $u(t,\cdot)$ is $\mathcal{F}_t$-measurable for a.a. t. Changing the process u in a set (in t) of zero measure, we obtain that u is adapted to $\{\mathcal{F}_t\}$.

(Proof-Cont.)
We have that

$$\lim_{n\to\infty} \mathbb{E}\left[(F_n - F)^2\right] = \lim_{n\to\infty} \mathbb{E}\left(\mathbb{E}[F_n] + \int_0^T u_s^{(n)} dB_s - F\right)^2 = 0.$$

On the other hand, by Itô isometry, we have

$$\lim_{n\to\infty} \mathbb{E}\left(\mathbb{E}[F_n] - \mathbb{E}[F]\right)^2 = 0$$

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T \left(u_s^{(n)} - u_s\right) dB_s\right)^2 = \lim_{n\to\infty} \mathbb{E}\int_0^T \left(u_s^{(n)} - u_s\right)^2 ds = 0.$$

and therefore $F = \mathbb{E}[F] + \int_0^T u_s dB_s$.

(Proof-Cont.)

3 Uniqueness: Assume that $u^{(1)}$ and $u^{(2)} \in L^2_{a,T}$ and

$$F = \mathbb{E}[F] + \int_0^T u_s^{(1)} dB_s = \mathbb{E}[F] + \int_0^T u_s^{(2)} dB_s.$$

By Itô isometry:

$$\mathbb{E}\left[\left(\int_0^T \left(u_s^{(1)} - u_s^{(2)}\right) dB_s\right)^2\right] = \mathbb{E}\left[\int_0^T \left(u_s^{(1)} - u_s^{(2)}\right)^2 ds\right] = 0.$$

Hence

$$u^{(1)}(t,\omega) = u(t,\omega)^{(2)} \quad \text{p.q.t.} \quad (t,\omega) \in [0,T] \times \Omega.$$

□

Martingale Representation Theorem

Theorem

(Martingale Rep. Theorem): Assume that $\{M_t, t \in [0,T]\}$ is a $\{\mathcal{F}_t\}$-martingale and $\mathbb{E}\left[M_T^2\right] < \infty$. Then, exists one and only one process $u \in L^2_{a,T}$ such that

$$M_t = \mathbb{E}[M_0] + \int_0^t u_s dB_s \qquad \forall t \in [0,T].$$

Proof: Apply the Itô Representation Theorem to $F = M_T$. Then $\exists^1 u \in L^2_{a,T}$ such that

$$M_T = \mathbb{E}[M_T] + \int_0^T u_s dB_s.$$

(Proof-Cont.)
Since $\{M_t, t \in [0, T]\}$ is a martingale, $\mathbb{E}[M_T] = \mathbb{E}[M_0]$ and

$$M_t = \mathbb{E}[M_T|\mathcal{F}_t] = \mathbb{E}[\mathbb{E}[M_T]|\mathcal{F}_t] + \mathbb{E}\left[\int_0^T u_s dB_s|\mathcal{F}_t\right]$$
$$= \mathbb{E}[M_0] + \int_0^t u_s dB_s.$$

where we have used the martingale property of the indefinite stochastic integral. □

Example

Let $F = B_T^3$. What is the Itô representation of F? By the Itô formula (applied to $f(x) = x^3$ and $B_T^3 = f(B_t)$), we obtain

$$B_T^3 = \int_0^T 3B_t^2 dB_t + 3\int_0^T B_t dt.$$

Integrating by parts, we have

$$\int_0^T B_t dt = TB_T - \int_0^T t dB_t = \int_0^T (T-t)\, dB_t.$$

Therefore

$$F = B_T^3 = \int_0^T 3\left[B_t^2 + (T-t)\right] dB_t. \quad (4)$$

since $\mathbb{E}\left[B_T^3\right] = 0$ (remember that $B_T \sim N(0, T)$).

Example

What is the process u such that $\int_0^T tB_t^2 dt - \frac{T^2}{2} B_T^2 = -\frac{T^3}{6} + \int_0^T u_t dB_t$?
Applying the Itô formula to $X_t = f(t, B_t) = t^2 B_t^2$, with $f(t,x) = t^2 x^2$, we have

$$T^2 B_T^2 = \int_0^T 2tB_t^2 dt + \int_0^T 2t^2 B_t dB_t + \int_0^T t^2 dt.$$

Therefore

$$\int_0^T tB_t^2 dt - \frac{T^2}{2} B_T^2 = -\frac{T^3}{6} - \int_0^T t^2 B_t dB_t.$$

Hence

$$u_t = -t^2 B_t.$$

Note that $\mathbb{E}\left[\int_0^T tB_t^2 dt - \frac{T^2}{2} B_T^2\right] = -\frac{T^3}{6}$.

Integration by parts formula

In general, the integration by parts formula is the following one.

Theorem

(integration by parts): Suppose that $f(s)$ is a deterministic function, continuous and of class C^1. Then, we have

$$\int_0^t f(s)\, dB_s = f(t)\, B_t - \int_0^t f'(s)\, B_s ds.$$

One can prove this formula by application of the Itô formula to $g(t,x) = f(t)\, x$. That is:

$$f(t)\, B_t = \int_0^t f'(s)\, B_s ds + \int_0^t f(s)\, dB_s.$$