Stochastic Calculus - Part 10

ISEG

Existence and Uniqueness Theorem for SDE's

- Let $T > 0$, $b(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(\cdot, \cdot) :$ $[0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be measurable functions such that:
  1) $\mathbb{E}\left[|Z|^2\right] < \infty$ and Z independent of B.
  2) Linear growth property
  $$|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|), \ x \in \mathbb{R}^n, \ t \in [0, T]$$
  3) Lipschitz property
  $$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq D|x - y|, \ x, y \in \mathbb{R}^n, \ t \in [0, T]$$
  Then the SDE
  $$X_t = Z + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s \tag{1}$$
  has a unique solution. Exists a unique stoch. proc. $X = \{X_t, 0 \leq t \leq T\}$ continuous, adapted, which satisfies (1) and
  $$E\left[\int_0^T |X_s|^2\, ds\right] < \infty.$$

Proof of the existence and uniqueness theorem

- Consider the space $L^2_{a,T}$ of processes adapted to the filtration $\mathcal{F}_t^Z := \sigma(Z) \cup \mathcal{F}_t$ such that $E\left[\int_0^T |X_s|^2 \, ds\right] < \infty$.
- In this space, consider the norm:

$$\|X\| = \left(\int_0^T e^{-\lambda s} E\left[|X_s|^2\right] ds\right)^{\frac{1}{2}},$$

where $\lambda > 2D^2(T+1)$.
- Define the operator $\mathcal{L} : L^2_{a,T} \to L^2_{a,T}$ by:

$$(\mathcal{L}X)_t = Z + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$$

Existence and uniqueness Theorem for SDEs

Proof of the theorem

- By the linear growth of b and σ, the operator $\mathcal{L}$ is well defined.
- By the Cauchy-Schwarz inequality and by Itô isometry, we have:

$$E\left[\left|(\mathcal{L}X)_t - (\mathcal{L}Y)_t\right|^2\right] \leq 2E\left[\left(\int_0^t (b(s, X_s) - b(s, Y_s))\, ds\right)^2\right]$$

$$+ 2E\left[\left(\int_0^t (\sigma(s, X_s) - \sigma(s, Y_s))\, dB_s\right)^2\right]$$

$$\leq 2TE\left[\int_0^t (b(s, X_s) - b(s, Y_s))^2\, ds\right] +$$

$$+ 2E\left[\int_0^t (\sigma(s, X_s) - \sigma(s, Y_s))^2\, ds\right]$$

Proof of the theorem

- By the Lipschitz property, we have:
$$E\left[\left|(\mathcal{L}X)_t-(\mathcal{L}Y)_t\right|^2\right]\leq 2D^2\,(T+1)\,E\left[\int_0^t (X_s-Y_s)^2\,ds\right].$$
- Define $K=2D^2\,(T+1)$. Multiplying the previous inequality by $e^{-\lambda t}$ and integrating in $[0,T]$, we have
$$\int_0^T e^{-\lambda t}E\left[\left|(\mathcal{L}X)_t-(\mathcal{L}Y)_t\right|^2\right]dt$$
$$\leq K\int_0^T e^{-\lambda t}E\left[\int_0^t (X_s-Y_s)^2\,ds\right]dt.$$
Interchanging the order of integration, we have
$$=K\int_0^T\left[\int_s^T e^{-\lambda t}dt\right]E\left[(X_s-Y_s)^2\right]ds$$
$$\leq\frac{K}{\lambda}\int_0^T e^{-\lambda s}E\left[(X_s-Y_s)^2\right]ds$$

Proof of the theorem

- Therefore
$$\|(\mathcal{L}X)-(\mathcal{L}Y)\|\leq\sqrt{\frac{K}{\lambda}}\,\|X-Y\|$$
- Choosing $\lambda>K$, we have $\sqrt{\frac{K}{\lambda}}<1$, and the operator $\mathcal{L}$ is a contraction in the space $L^2_{a,T}$. Hence, by the fixed point theorem, exists a unique fixed point to $\mathcal{L}$ and that fixed point is exactly the solution of the SDE:
$$(\mathcal{L}X)_t=X_t.$$
- See the book of Oksendal for a proof based on Picard approximations and the Gronwall inequality.

Examples

- The Geometric Brownian motion

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t\right]$$

We know that it is the solution of the SDE

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dB_t, \\ S_0 &= S_0. \end{aligned}$$

This SDE models the time evolution of the price of a risky financial asset in the standard Black-Scholes model.

Example

- Consider the Black-Scholes SDE with coefficients $\mu(t)$ and $\sigma(t) > 0$ depending on time:

$$\begin{aligned} dS_t &= S_t\left(\mu(t)\, dt + \sigma(t)\, dB_t\right), \\ S_0 &= S_0. \end{aligned}$$

- How is the solution of this SDE?

Example

- Let $S_t = \exp(Z_t)$ and $Z_t = \ln(S_t)$. By Itô formula with $f(x) = \ln(x)$, we have:

$$dZ_t = \frac{1}{S_t}\left(S_t\left(\mu(t)\,dt + \sigma(t)\,dB_t\right)\right) - \frac{1}{2S_t^2}\left(S_t^2\sigma^2(t)\,dt\right)$$
$$= \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)\,dB_t.$$

  Hence,

$$Z_t = Z_0 + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)\,dB_s.$$

- Therefore,

$$S_t = S_0 \exp\left(\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)\,dB_s\right).$$

Orsntein-Uhlenbeck process with mean reversion

$$dX_t = a(m - X_t)\,dt + \sigma dB_t,$$
$$X_0 = x.$$

$a, \sigma > 0$ and $m \in \mathbb{R}$.

- Solution of the associated homogeneous ODE $dx_t = -ax_t dt$ is $x_t = xe^{-at}$.
- Consider that the process is such that $X_t = Y_t e^{-at}$ or $Y_t = X_t e^{at}$.
- By the Itô formula applied to $f(t, x) = xe^{at}$, we have

$$Y_t = x + m\left(e^{at} - 1\right) + \sigma \int_0^t e^{as}\,dB_s.$$

Orsntein-Uhlenbeck process with mean reversion

- Hence,
$$X_t = m + (x - m)\, e^{-at} + \sigma e^{-at} \int_0^t e^{as} dB_s.$$
- This is a Gaussian process, since it is a stochastic integral of the type $\int_0^t f(s)\, dB_s$, where f is a deterministic function.
- Mean:
$$E[X_t] = m + (x - m)\, e^{-at}$$

Ornstein-Uhlenbeck process with mean reversion:

- Covariance: by Itô isometry
$$\begin{aligned} \mathrm{Cov}[X_t, X_s] &= \sigma^2 e^{-a(t+s)} E\left[\left(\int_0^t e^{ar} dB_r\right)\left(\int_0^s e^{ar} dB_r\right)\right] \\ &= \sigma^2 e^{-a(t+s)} \int_0^{t\wedge s} e^{2ar} dr \\ &= \frac{\sigma^2}{2a}\left(e^{-a|t-s|} - e^{-a(t+s)}\right). \end{aligned}$$

Note that
$$X_t \sim N\left[m + (x - m)\, e^{-at}, \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)\right].$$

Ornstein-Uhlenbeck with mean reversion:

- When $t \to \infty$, the distribution of X_t converges to

$$\nu := N\left[m, \frac{\sigma^2}{2a}\right].$$

  which is the invariant or stationary distribution.
- Note that if X_0 has distribution ν then X_t has the same distribution ν for all t.

Financial applications of the Ornstein-Uhlenbeck process with mean reversion

- Vasicek model for the interest rate

$$dr_t = a\,(b - r_t)\,dt + \sigma dB_t,$$

  with a, b, σ parameters.
- Solution:

$$r_t = b + (r_0 - b)\,e^{-at} + \sigma e^{-at}\int_0^t e^{as}dB_s.$$

Financial applications of the Ornstein-Uhlenbeck process with mean reversion:

- Black-Scholes model with stochastic volatlity: consider that the volatility $\sigma(t) = f(Y_t)$ is a function of a Ornstein-Uhlenbeck process with mean reversion.

$$dY_t = a(m - Y_t)\,dt + \beta dW_t,$$

  with a, m, β parameters and where $\{W_t, 0 \leq t \leq T\}$ is a Brownian motion.
- The SDE that models the time evolution of the price of the risky asset is

$$dS_t = \mu S_t dt + f(Y_t)\,S_t dB_t$$

  where $\{B_t, 0 \leq t \leq T\}$ is a Brownian motion.and the Brownian motions W_t and B_t may be correlated, i.e.,

$$E[B_t W_s] = \rho(s \wedge t).$$

Example

- Consider the SDE

$$X_t = x + \int_0^t f(s, X_s)\,ds + \int_0^t c(s)\,X_s dB_s,$$

  where f and c are continuous deterministic functions and f satisfies the Lipschitz and linear growth conditions in x.
- By the existence and uniqueness theorem for SDE's, exists one unique solution for this SDE.
- How can we obtain the solution?

Example

- Consider the "integrating factor"
$$F_t = \exp\left(\int_0^t c(s)\, dB_s - \frac{1}{2}\int_0^t c(s)^2\, ds\right).$$
Note that F_t is a solution of the SDE if $f = 0$ and $x = 1$.
- Suppose that $X_t = F_t Y_t$ or that $Y_t = (F_t)^{-1} X_t$. Then, by Itô formula,
$$dY_t = (F_t)^{-1} f(t, F_t Y_t)\, dt$$
and $Y_0 = x$.
- This equation for Y is a ODE with random coefficients (is a deterministic ODE parametrized by $\omega \in \Omega$).

Example

- For example, if $f(t, x) = f(t)x$, then we have the ODE
$$\frac{dY_t}{dt} = f(t) Y_t$$
and therefore
$$Y_t = x \exp\left(\int_0^t f(s)\, ds\right).$$
Hence
$$X_t = x \exp\left(\int_0^t f(s)\, ds + \int_0^t c(s)\, dB_s - \frac{1}{2}\int_0^t c(s)^2\, ds\right).$$

Linear SDE's

- In general, a linear SDE has the form:

$$
\begin{aligned}
dX_t &= (a(t) + b(t) X_t)\, dt + (c(t) + d(t) X_t)\, dB_t, \\
X_0 &= x,
\end{aligned}
$$

  where a, b, c, d are deterministic continuous functions.
- How to obtain the solution of the SDE?

Linear SDE's

- Assume that

$$X_t = U_t V_t, \tag{2}$$

  where

$$
\begin{cases}
dU_t = b(t) U_t dt + d(t) U_t dB_t, \\
dV_t = \alpha(t)\, dt + \beta(t)\, dB_t.
\end{cases}
$$

  and $U_0 = 1,\ V_0 = x$.
- From a previous example, we know that

$$U_t = \exp\left(\int_0^t b(s)\, ds + \int_0^t d(s)\, dB_s - \frac{1}{2} \int_0^t d(s)^2\, ds \right) \tag{3}$$

Linear SDE's

- On the other hand, calculating the differential of (2), by Ito's formula with $f(u,v) = uv$, we have

$$\begin{aligned} dX_t &= V_t dU_t + U_t dV_t + \frac{1}{2}(dU_t)(dV_t) + \frac{1}{2}(dV_t)(dU_t) \\ &= (b(t)X_t + \alpha(t)U_t + \beta(t)d(t)U_t)\,dt + (d(t)X_t + \beta(t)U_t)\,dB_t. \end{aligned}$$

- Comparing with the initial SDE for X, we have that

$$\begin{aligned} a(t) &= \alpha(t)U_t + \beta(t)d(t)U_t, \\ c(t) &= \beta(t)U_t. \end{aligned}$$

Linear SDE's

Linear SDE's

- Hence

$$\begin{aligned} \beta(t) &= c(t)U_t^{-1}, \\ \alpha(t) &= [a(t) - c(t)d(t)]\,U_t^{-1}. \end{aligned}$$

- Therefore,

$$X_t = U_t\left(x + \int_0^t [a(s) - c(s)d(s)]\,U_s^{-1}ds + \int_0^t c(s)\,U_s^{-1}dB_s\right),$$

where U_t is given by (3).

SDE's - Theorem of existence and uniqueness for the one-dimensional case

- In the one-dimensional case ($n = 1$), the Lipschitz condition for σ in the existence and uniqueness theorem can be weakened if $\sigma(t, x) = \sigma(x)$, $b(t, x) = b(x)$ (coefficients do not depend on time).
- Assume that b satisfies the Lipschitz condition and the coefficient σ satisfies the condition

$$|\sigma(t, x) - \sigma(t, y)| \leq D |x - y|^{\alpha}, \ x, y \in \mathbb{R}, \ t \in [0, T]$$

  with $\alpha \geq \frac{1}{2}$. Then, exists one unique solution for the SDE.
- As an example, the SDE for the Cox-Ingersoll-Ross (CIR) model for interest rates

$$\begin{aligned} dr_t &= a(b - r_t)\, dt + \sigma \sqrt{r_t} dB_t \\ r_0 &= x, \end{aligned}$$

  has one and only one solution.

Exercise

- The Cox-Ingersoll-Ross (CIR) model for the interest rate $R(t)$ is given by

$$dR(t) = (\alpha - \beta R(t))\, dt + \sigma \sqrt{R(t)}\, dW(t),$$

  where α, β and σ are positive constants. The CIR equation does not have a solution in closed form. However, one can find the mean and the variance of $R(t)$.

a) Calculate the mean value of $R(t)$. (Hint: Let $X(t) = e^{\beta t} R(t)$ and apply the It formula).
b) Calculate the variance of $R(t)$. (Hint: Calculate $d\left(X^2(t)\right)$ using the Itô formula in the differential form and integrate).
c) Calculate $\lim_{t \to +\infty} Var(R(t))$.