Stochastic Calculus - part 12

ISEG

The Markov property for diffusion processes

- The solutions of SDE's are called Diffusion Processes.
- Let $X = \{X_t, t \geq 0\}$ be a diffusion process (of dimension n) which satisfies the SDE

$$dX_t = b\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dB_t, \tag{1}$$

where B is a m-dimensional Brownian motion and b and σ satisfy the conditions of the existence and uniqueness theorem.

The Markov property for diffusion processes

Definition

We say that a process $X = \{X_t, t \geq 0\}$ *is a Markov process if* $\forall s < t$*, we have that*

$$E\left[f\left(X_t\right)|X_r, r \leq s\right] = E\left[f\left(X_t\right)|X_s\right].$$

for any bounded and measurable function f *defined on* $\mathbb{R}^n$*.*

- In particular, if $C \subset \mathbb{R}^n$ is measurable, then

$$P\left[X_t \in C|X_r, r \leq s\right] = P\left[X_t \in C|X_s\right].$$

- Markov property: the future values of a process depend only of its present value and not from its past values (if the present value is known).
- The probability law for Markov processes is described by the transition probabilities:

$$P\left(C, t, x, s\right) := P\left(X_t \in C|X_s = x\right), \quad 0 \leq s < t.$$

The Markov property for diffusion processes

- $P(\cdot, t, x, s)$ is the law of probability of X_t conditional to $X_s = x$. if this conditional probability has an associated density, we represent it by
$$p(y, t, x, s).$$

Example

The Brownian motion is a Markov process with transition probabilities

$$p(y, t, x, s) = \frac{1}{\sqrt{2\pi (t-s)}} \exp\left(-\frac{(x-y)^2}{2(t-s)} \right). \tag{2}$$

Indeed,

$$\begin{aligned} P[B_t \in C | \mathcal{F}_s] &= P[B_t - B_s + B_s \in C | \mathcal{F}_s] \\ &= P[B_t - B_s + x \in C] |_{x=B_s} \\ &= P[B_t \in C | B_s = x], \end{aligned}$$

where we have used the properties of conditional probability and the fact that $B_t - B_s$ is independent of $\mathcal{F}_s$ and B_s is known if we know the "information" $\mathcal{F}_s$ (that is, B_s is $\mathcal{F}_s$-measurable).
Since $B_t - B_s + x$ has normal distribution with mean x and variance $t - s$, then the density of the transition probability is given by (2).

The Markov property for diffusion processes

- Notation: $\{X_t^{s,x}, t \geq s\}$ is the solution of the SDE (1) defined in $[s, +\infty)$ and with initial condition $X_s^{s,x} = x$.
- If $s = 0$, we use the notation $X_t^{0,x} = X_t^x$.
- Properties
  1. Exists a continuous version (in all the parameters s, t, x) of the process $\{X_t^{s,x}, 0 \leq s \leq t, x \in \mathbb{R}^n\}$.
  2. For any $t \geq s$, we have
$$X_t^x = X_t^{s,X_s^x}. \tag{3}$$

The Markov property for diffusion processes

Theorem

(Markov property for diffusion processes): Let f be a bounded function in $\mathbb{R}^n$. Then, for any $0 \leq s \leq t$, we have
$$E\left[f\left(X_t^x\right)|\mathcal{F}_s\right] = E\left[f\left(X_t^{s,y}\right)\right]|_{y=X_s} \tag{4}$$

- The diffusion processes are Markov processes.

The Markov property for diffusion processes

- The transition probabilities for a diffusion process are the probabilities

$$P(C, t, x, s) = P(X_t^{s,x} \in C).$$

- If a diffusion process is homogeneous in time (the coefficients b and σ do not depend on t) then the Markov property (4) can be written as

$$E\left[f\left(X_t^x\right)|\mathcal{F}_s\right] = E\left[f\left(X_{t-s}^y\right)\right]|_{y=X_s^x}.$$

The Markov property for diffusion processes

Exercise: Calculate the transition probabilities of the Ornstein-Uhlenbeck process with mean reversion.

- Solution:

$$dX_t = a\left(m - X_t\right)dt + \sigma dB_t.$$

Solution in $[s, +\infty)$, with initial condition $X_s = x$:

$$X_t^{s,x} = m + (x - m)\, e^{-a(t-s)} + \sigma e^{-at} \int_s^t e^{ar} dB_r.$$

We know that $\left\{\int_s^t e^{ar} dB_r, t \geq s\right\}$ is a Gaussian process with mean and variance given by (see previous lecture)

$$E\left[X_t^{s,x}\right] = m + (x - m)\, e^{-a(t-s)},$$

$$\mathrm{Var}\left[X_t^{s,x}\right] = \frac{\sigma^2}{2a}\left(1 - e^{-2a(t-s)}\right).$$

The Markov property for diffusion processes

- (solution) The transition probability

$$P(\cdot, t, x, s) \text{ is obtained from the Distribution of } X_t^{s,x}.$$

Therefore, it can be calculated from the normal distribution with the mean and variance presented before.

The Stratovonich integral and SDE's

- In the Itô stochastic integral for simple processes, when we define $\int_0^t u_s dB_s$ using sums of the Riemann type, we use always the value of u at point t_{j-1} and assume that the process is constant in $[t_{j-1}, t_j)$.
- As a consequence, the expected value of the Itô integral is zero and its variance can be calculated by the Itô isometry. Moreover, the undefined Itô integral is a Martingale.
- The drawback of the Itô integral is that in the "chain rule" (or Itô formula) we have an unusual term of 2nd order (term that does not appear in the chain rule of classical calculus).

The Stratovonich integral and SDE's

- The Stratonovich integral $\int_0^T u_s \circ dB_s$ is defined as the limit in probability of the sequence of

$$\sum_{i=1}^{n} \frac{1}{2} \left(u_{t_{i-1}} + u_{t_i} \right) \Delta B_i,$$

where $t_i = \frac{iT}{n}$.

The Stratovonich integral and SDE's

- Relationship between the Itô integral and the Stratonovich integral: If u is a Itô process of the form

$$u_t = u_0 + \int_0^t \beta_s ds + \int_0^t \alpha_s dB_s, \tag{5}$$

then it is possible to show that

$$\int_0^T u_s \circ dB_s = \int_0^T u_s dB_s + \frac{1}{2} \int_0^T \alpha_s ds.$$

The Stratovonich integral and SDE's

- The Itô formula for the stochastic integral of Stratonovich is the usual chain rule used in classical calculus.
- Indeed, if u is a process of the form (5) and

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s \circ dB_s$$

then one can show that

$$df\left(X_t\right) = f'\left(X_t\right) \circ dX_t$$

The Stratovonich integral and SDE's

- A SDE in the sense of Itô can be transformed in a SDE in the sense of Stratonovich, using the formula that relates both integrals
- Itô SDE:

$$X_t = X_0 + \int_0^t b\left(s, X_s\right) ds + \int_0^t \sigma\left(s, X_s\right) dB_s.$$

- Equivalent Stratonovich SDE:

$$X_t = X_0 + \int_0^t b\left(s, X_s\right) ds - \frac{1}{2}\int_0^t \left(\sigma\sigma'\right)\left(s, X_s\right) ds + \int_0^t \sigma\left(s, X_s\right) \circ dB_s.$$

- This is a simple consequence of the Itô decomposition of $\sigma\left(t, X_t\right)$, which is

$$\sigma\left(t, X_t\right) = \sigma\left(0, X_0\right) + \int_0^t h_s ds + \int_0^t \left(\sigma\sigma'\right)\left(s, X_s\right) dB_s.$$