Stochastic Calculus - part 13

ISEG

Infinitesimal generator of a diffusion

- Consider a one n-dimensional diffusion X that satisfies the SDE

$$\begin{aligned} dX_t &= b\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dB_t, \\ X_0 &= x_0. \end{aligned}$$

- Assume that b and σ satisfy the conditions of the existence and uniqueness theorem of SDE's, $b : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \to M\left(n, m\right)$, $M\left(n, m\right)$ is the set of $n \times m$ matrices, $x_0 \in \mathbb{R}^n$.

Definition

The infinitesimal generator associated to the diffusion X is the differential operator of 2nd order A defined by

$$Ah\left(t, x\right) := \sum_{i=1}^{n} b_i\left(t, x\right) \frac{\partial h}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \left(\sigma\sigma^T\right)_{i,j}\left(t, x\right) \frac{\partial^2 h}{\partial x_i \partial x_j},$$

where h is a $C^{1,2}$ function defined on $\mathbb{R}^+ \times \mathbb{R}^n$.

- The infinitesimal generator is also called Dynkin operator, Itô operator or Kolmogorov Backward operator.
- Relationship between the diffusion X and the operator A: By Itô formula, if $f(t,x)$ is a $C^{1,2}$ function, then $f(t,X_t)$ is an Itô process such that:

$$df(t,X_t) = \left\{ \frac{\partial f}{\partial t}(t,X_t) + Af(t,X_t) \right\} dt + [\nabla_x f(t,X_t)] \sigma(t,X_t)\, dB_t, \tag{1}$$

where the gradient is defined by

$$\nabla_x f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right].$$

- Note that if

$$E \int_0^t \left(\frac{\partial f}{\partial x_i}(t,X_t)\, \sigma_{i,j}(t,X_t) \right)^2 ds < \infty, \tag{2}$$

for all $t > 0$ and for all i, j, then all the stochastic integrals in (1) are well defined and are martingales. Therefore

$$M_t = f(t,X_t) - \int_0^t \left(\frac{\partial f}{\partial s}(s,X_s) + Af(s,X_s) \right) ds$$

is a martingale.
- A sufficient condition for (2) to be satisfied is that the partial derivatives of $f(s,X_s)$ have linear growth, i.e.

$$\left| \frac{\partial f}{\partial x_i}(t,x) \right| \leq C(1 + |x|).$$

PDE's

- The PDE

$$\frac{\partial F}{\partial t}(t,x) + AF(t,x) = 0, \qquad (3)$$
$$F(T,x) = \Phi(x)$$

is a parabolic PDE with a terminal condition (in T).
- This PDE can also be written (assuming that $n = 1$, for a simpler notation)

$$\frac{\partial F}{\partial t}(t,x) + b(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} = 0, \qquad (4)$$
$$F(T,x) = \Phi(x).$$

PDE's

- Instead of solving the PDE analytically, we will try to obtain a solution, using a "stochastic representation formula"
- Assume that exists a solution F. Let us fix t and x and define the process X in $[t, T]$ as the solution of the SDE

$$dX_s = b(s, X_s)\,ds + \sigma(s, X_s)\,dB_s,$$
$$X_t = x.$$

- The infinitesimal generator associated to X is

$$A = b(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2},$$

which is exactly the differential operator in (3) or (4).

- Applying the Itô formula to F, we obtain (see (1)):

$$F(T, X_T) = F(t, X_t) + \int_t^T \left(\frac{\partial F}{\partial s}(s, X_s) + AF(s, X_s) \right) ds + \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)\, dB_s.$$

- We know that $\frac{\partial F}{\partial s}(s, X_s) + AF(s, X_s) = 0$ and applying the expected values (considering the initial value $X_t = x$), we obtain

$$E_{t,x}[F(T, X_T)] = E_{t,x}[F(t, X_t)].$$

- Since, by the terminal values (or boundary values), $E_{t,x}[F(T, X_T)] = E_{t,x}\left[\Phi(X_T^{t,x})\right]$ and $E_{t,x}\left[F\left(t, X_t^{t,x}\right)\right] = F(t, x)$, we have that

$$F(t, x) = E_{t,x}\left[\Phi(X_T^{t,x})\right],$$

and this is a "stochastic representation formula" for the solution of the PDE (4).

Feynman-Kac Formula

Proposition

Assume that F is a solution of the problem (4). Assume that $\sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)$ is a process in L^2 (i.e. $E\int_0^t \left(\frac{\partial f}{\partial x_i}(t, X_t)\,\sigma_{i,j}(t, X_t)\right)^2 ds < \infty$). Then

$$F(t, x) = E_{t,x}\left[\Phi(X_T^{t,x})\right],$$

where $X_s^{t,x}$ satisfies

$$dX_s = b(s, X_s)\,ds + \sigma(s, X_s)\,dB_s,$$
$$X_t = x.$$

Feynman-Kac Formula (multidimensional case)

Proposition

Assume that F is a solution of problem (3). Assume that $E\int_0^t \left(\frac{\partial f}{\partial x_i}(t, X_t)\,\sigma_{i,j}(t, X_t)\right)^2 ds < \infty$, for all $t > 0$ and for all i, j. Then

$$F(t, x) = E_{t,x}\left[\Phi(X_T^{t,x})\right],$$

where $X_s^{t,x}$ satisfies

$$dX_s = b(s, X_s)\,ds + \sigma(s, X_s)\,dB_s,$$
$$X_t = x.$$

Notes on PDE's

- A parabolic PDE is a PDE of 2nd order of the type

$$Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0,$$

where $B^2 - 4AC = 0$.
- Example: the "heat equation" in dimension one:

$$u_t = ku_{xx}.$$

A more general Feynman-Kac formula

- Consider that the function $q(x)$ is continuous and lower bounded, with $q \in C(\mathbb{R}^n)$.
- Consider that the PDE

$$\frac{\partial F}{\partial t}(t,x) + AF(t,x) - q(x)F(t,x) = 0, \quad (5)$$
$$F(T,x) = \Phi(x)$$

with boundary terminal condition (in T).
- The previous PDE can also be written as (assuming $n = 1$, for a simpler notation)

$$\frac{\partial F}{\partial t}(t,x) + b(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} - q(x)F(t,x) = 0, \quad (6)$$
$$F(T,x) = \Phi(x).$$

A more general Feynman-Kac formula

- Instead of solving the PDE in an analytic way, we will try to obtain a "stochastic representation formula" for the solution.
- Assume that exists a solution F.
  Let us fix t and x and define the process X in $[t, T]$ as the solution of the SDE

$$dX_s = b(s, X_s)\, ds + \sigma(s, X_s)\, dB_s,$$
$$X_t = x.$$

- The infinitesimal generator associated to X is

$$A = b(t, x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2}{\partial x^2},$$

which is exactly the operator in PDE (5) or (6).

A more general Feynman-Kac formula

- Applying the Itô formula to $g(t, X_t) = \exp\left(-\int_0^t q(X_s)\, ds\right) F(t, X_t)$ and integrating between t and T, we have

$$\exp\left(-\int_0^T q(X_s)\, ds\right) F(T, X_T) = \exp\left(-\int_0^t q(X_s)\, ds\right) F(t, X_t) +$$
$$+ \int_t^T e^{-\int_0^s q(X_r) dr} \left(\frac{\partial F}{\partial s}(s, X_s) + AF(s, X_s) - q(X_s) F(s, X_s) \right) ds$$
$$+ \int_t^T \exp\left(-\int_0^s q(X_r)\, dr\right) \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s)\, dB_s.$$

We have $\frac{\partial F}{\partial s}(s, X_s) + AF(s, X_s) - q(X_s) F(s, X_s) = 0$ and by the expected value (with $X_t = x$), we obtain

$$E_{t,x}\left[\exp\left(-\int_t^T q(X_s)\, ds\right) F(T, X_T)\right] = E_{t,x}\left[F(t, X_t)\right],$$

assuming that the stochastic integral is well defined and that therefore, its expected value is zero.

A more general Feynman-Kac formula

- It is clear that $E_{t,x}\left[\exp\left(-\int_t^T q\left(X_s\right)ds\right)F\left(T,X_T\right)\right] = E_{t,x}\left[\exp\left(-\int_t^T q\left(X_s\right)ds\right)\Phi(X_T^{t,x})\right]$ and $E_{t,x}\left[F\left(t,X_t^{t,x}\right)\right] = F\left(t,x\right)$. Therefore

$$F\left(t,x\right) = E_{t,x}\left[\exp\left(-\int_t^T q\left(X_s^{t,x}\right)ds\right)\Phi(X_T^{t,x})\right],$$

and this is the stochastic representation formula for the solution of PDE (5) ou (6).

Feynman-Kac Formula 2

Proposition

Let F be a solution of problem (5) ou (6). Assume that $\sigma\left(s,X_s\right)\frac{\partial F}{\partial x}\left(s,X_s\right)$ is a process in $L^2_{a,T}$ (i.e. $E\int_0^T\left[\frac{\partial F}{\partial x}\left(s,X_s\right)\sigma\left(s,X_s\right)\right]^2 ds < \infty$). Then

$$F\left(t,x\right) = E_{t,x}\left[\exp\left(-\int_t^T q\left(X_s^{t,x}\right)ds\right)\Phi(X_T^{t,x})\right],$$

where $X_s^{t,x}$ satisfies

$$dX_s = b\left(s,X_s\right)ds + \sigma\left(s,X_s\right)dB_s,$$
$$X_t = x.$$

- Note: Assuming that $q(x)$ is a continuous and lower bounded function, a sufficient condition for $E\int_0^T \left[\exp\left(-\int_0^s q\left(X_r\right)dr\right)\frac{\partial F}{\partial x}\left(s,X_s\right)\sigma\left(s,X_s\right)\right]^2 ds<\infty$ is that the derivative $\frac{\partial F}{\partial x}\left(s,x\right)$ has linear growth, i.e.

$$\left|\frac{\partial F}{\partial x}\left(s,x\right)\right|\leq C\left(1+\left|x\right|\right).$$