Stochastic Calculus - part 15

ISEG

Girsanov Theorem - What is it?

- The Girsanov theorem states, in it's simpler version, that the Brownian motion with drift: $\widetilde{B}_t = B_t + \lambda t$, may be seen as a standard Brownian motion if we change the probability measure.
- In a broader way, the theorem states that if we change the drift coefficient of an It process then the law of the process does not radically change.

Changing the probability measure

- Assume that $L \geq 0$ is a random variable with mean 1 defined on the probab. space $(\Omega, \mathcal{F}, P)$. Then

$$Q(A) = E\left[\mathbf{1}_A L\right]$$

defines a new probability measure It is clear that $Q(\Omega) = E\left[L\right] = 1$.

- $Q(A) = E\left[\mathbf{1}_A L\right]$ is equivalent to

$$\int_\Omega \mathbf{1}_A dQ = \int_\Omega \mathbf{1}_A L dP.$$

- We say that L is the density of Q with respect to P and is written

$$\frac{dQ}{dP} = L.$$

- L is also the Radon-Nikodym of Q with respect to P.

Changing the probability measure

- The expected value of a r.v. X defined in the probability space $(\Omega, \mathcal{F}, P)$ is calculated by the formula

$$E_Q\left[X\right] = E\left[XL\right].$$

- The probability measure Q is absolutely continuous with respect to P, which means that

$$P(A) = 0 \Longrightarrow Q(A) = 0.$$

- If the random variable L is strictly positive $(L > 0)$, the measures P and Q are equivalent (that is, they are mutually absolutely continuous), which means that

$$P(A) = 0 \Longleftrightarrow Q(A) = 0.$$

Example - Simple Version of Girsanov Theorem

- Let $X \sim N\left(m, \sigma^2\right)$. Is there a probability measure Q with respect to which $X \sim N\left(0, \sigma^2\right)$?
- Consider the r.v.

$$L = \exp\left(-\frac{m}{\sigma^2}X + \frac{m^2}{2\sigma^2}\right).$$

- It is easily verified that $E\left[L\right] = 1$. Consider the density of the normal distribution $N\left(m, \sigma^2\right)$ and it follows that

$$\begin{aligned} E\left[L\right] &= \int_{-\infty}^{+\infty} \exp\left(-\frac{m}{\sigma^2}x + \frac{m^2}{2\sigma^2}\right) \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = 1. \end{aligned}$$

Example - Simple version of the Girsanov theorem

- Assume that Q has density L with respect to P. Then, in $(\Omega, \mathcal{F}, Q)$, X has the characteristic function:

$$\begin{aligned} E_Q\left[e^{itX}\right] &= E\left[e^{itX}L\right] \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(itx - \frac{m}{\sigma^2}x + \frac{m^2}{2\sigma^2}\right) \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right) dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(itx - \frac{x^2}{2\sigma^2}\right) dx = e^{-\frac{\sigma^2 t^2}{2}}. \end{aligned}$$

Conclusion: $X \sim N\left(0, \sigma^2\right)$.

Girsanov Theorem - 1st version

- $\{B_t, t \in [0, T]\}$ is a Brownian motion.
- Fix a real number λ and consider the martingale

$$L_t = \exp\left(-\lambda B_t - \frac{\lambda^2}{2} t\right). \tag{1}$$

- Exercise: Prove that $\{L_t, t \in [0, T]\}$ is a positive martingale with expected value 1 and satisfies the SDE

$$L_t = 1 - \int_0^t \lambda L_s dB_s$$

Girsanov Theorem - 1st version

- A r.v. $L_T = \exp\left(-\lambda B_T - \frac{\lambda^2}{2} T\right)$ is a density in $(\Omega, \mathcal{F}_T, P)$, and we can define a new probability measure

$$Q(A) = E[\mathbf{1}_A L_T],$$

  for each $A \in \mathcal{F}_T$.
- As $\{L_t, t \in [0, T]\}$ is a martingale, then $L_t = \exp\left(-\lambda B_t - \frac{\lambda^2}{2} t\right)$ is a density in $(\Omega, \mathcal{F}_t, P)$ and the probability measure Q has the density L_t.
- In fact, if $A \in \mathcal{F}_t$ then

$$\begin{aligned} Q(A) &= E[\mathbf{1}_A L_T] = E[E[\mathbf{1}_A L_T | \mathcal{F}_t]] \\ &= E[\mathbf{1}_A E[L_T | \mathcal{F}_t]] = E[\mathbf{1}_A L_t], \end{aligned}$$

  by the conditional expectation properties and the martingale property of $\{L_t, t \in [0, T]\}$.

Girsanov Theorem - 1st version

Theorem

(Girsanov Theorem I): On the probability space $(\Omega, \mathcal{F}_T, Q)$, where Q is defined by $Q(A) = E[\mathbf{1}_A L_T]$, the stochastic process

$$\widetilde{B}_t = B_t + \lambda t$$

is a Brownian motion

Technical Lemma

- We need the following lemma.

Lemma

Suppose X is a real r.v. and that $\mathcal{G}$ is a σ-algebra such that:

$$E\left[e^{iuX}|\mathcal{G}\right] = e^{-\frac{u^2\sigma^2}{2}}.$$

Then the random variable X is independent from the σ-algebra $\mathcal{G}$ and has normal distribution $N\left(0, \sigma^2\right)$.

The proof of the above lemma may be found in the lecture notes of Nualart, pgs. 63-64.

Proof of the Girsanov theorem

Proof.

It suffices to show that in $(\Omega, \mathcal{F}_T, Q)$, the increment $\widetilde{B}_t - \widetilde{B}_s$, with $s < t \leq T$, is independent from $\mathcal{F}_s$ and has normal distribution $N(0, t-s)$.
Taking into account the previous lemma, the result follows from the relation:

$$E_Q\left[\mathbf{1}_A e^{iu\left(\widetilde{B}_t - \widetilde{B}_s\right)}\right] = Q(A)\, e^{-\frac{u^2}{2}(t-s)}, \tag{2}$$

for all $s < t$, $A \in \mathcal{F}_s$ and $u \in \mathbb{R}$. In fact, if (2) is verified, then, from the definition of conditional expectation and the previous lemma, $\left(\widetilde{B}_t - \widetilde{B}_s\right)$ is independent from $\mathcal{F}_s$ and has normal distribution $N(0, t-s)$.
Now, we only need to prove the equality (2). □

Proof of the Girsanov Theorem

Proof.

(contin.) Proof of the equality (2):

$$\begin{aligned} E_Q\left[\mathbf{1}_A e^{iu\left(\widetilde{B}_t - \widetilde{B}_s\right)}\right] &= E\left[\mathbf{1}_A e^{iu\left(\widetilde{B}_t - \widetilde{B}_s\right)} L_t\right] \\ &= E\left[\mathbf{1}_A e^{iu(B_t - B_s) + iu\lambda(t-s) - \lambda(B_t - B_s) - \frac{\lambda^2}{2}(t-s)} L_s\right] \\ &= E\left[\mathbf{1}_A L_s\right] E\left[e^{(iu-\lambda)(B_t - B_s)}\right] e^{iu\lambda(t-s) - \frac{\lambda^2}{2}(t-s)} \\ &= Q(A) e^{\frac{(iu-\lambda)^2}{2}(t-s) + iu\lambda(t-s) - \frac{\lambda^2}{2}(t-s)} \\ &= Q(A) e^{-\frac{u^2}{2}(t-s)}, \end{aligned}$$

Where the definition of E_Q and L_t, independence of $(B_t - B_s)$ from L_s and A, and the definition of Q were used. □

Girsanov Theorem - second version

Theorem

(Girsanov Theorem II): Let $\{\theta_t, t \in [0, T]\}$ *be an adapted stochastic process that satisfies the Novikov condition:*

$$E\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2 dt\right)\right] < \infty. \tag{3}$$

Then, the stochastic process

$$\widetilde{B}_t = B_t + \int_0^t \theta_s ds$$

is a Brownian motion with respect to the measure Q defined by $Q(A) = E[\mathbf{1}_A L_T]$*, where*

$$L_t = \exp\left(-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right).$$

- Note that L_t satisfies the linear SDE

$$L_t = 1 - \int_0^t \theta_s L_s dB_s.$$

- It is necessary, for the process L_t to be a density, that $E[L_t] = 1$. However, condition (3) is sufficient to guarantee that this is in fact verified.
- The second version of the Girsanov theorem generalizes the first: note that, taking $\theta_t \equiv \lambda$, we obtain the previous version.