Hotelling Rule

The problem is:

$$\begin{aligned} & \underset{Q(t),\ 0\leq t\leq T}{Max} \int_0^T [P(t)Q(t) - C]e^{-rt}dt \\ s.t\ S(0) &= S_0 \text{ given} \\ \dot{S}(t) &= -Q(t). \end{aligned}$$

where

- $Q(t)$ is the rate of extraction;
- $S(t)$ is the stock of the resource;
- C are the constant extraction costs.

Remark 1 *we assume that* $S(T) = 0$.

Then,

$$H(Q, S, q, t) = [P(t)Q(t) - C]e^{-rt} - q(t)Q(t),$$

where $q(t)$ is the shadow price of the stock in present value (i.e., in moment 0).

The necessary conditions for an optimum are:

$$\begin{aligned} \frac{\partial H}{\partial Q} &= 0 \Leftrightarrow P(t)e^{-rt} = q(t) \\ \dot{q} &= -\frac{\partial H}{\partial S_t} \Leftrightarrow \dot{q} = 0 \\ \dot{S}(t) &= -Q(t),\ S(0) = S_0. \end{aligned}$$

Alternatively, we can consider H in current values:

$$H^C(Q, S, q, t) = P(t)Q(t) - C - \mu(t)Q(t),$$

where $\mu(t) = q(t)e^{rt}$ is the shadow price of the stock in moment t.

The necessary conditions for an optimum are:

$$\begin{aligned} \frac{\partial H^C}{\partial Q} &= 0 \Leftrightarrow P(t) = \mu(t) \qquad (1) \\ \dot{\mu} &= r\mu(t) - \frac{\partial H^C}{\partial S_t} \Leftrightarrow \dot{\mu} = r\mu(t) \\ \dot{S}(t) &= -Q(t),\ S(0) = S_0. \end{aligned}$$

From (1) we have:

$$\frac{\dot{\mu}(t)}{\mu(t)} = r.$$

Since $P(t) = \mu(t)$,

$$\frac{\dot{P}(t)}{P(t)} = r,$$

the famous Hotelling rule. Integrating,

$$\begin{aligned}
\int_0^t \frac{\dot{P}(s)}{P(s)} ds &= \int_0^t r ds \\
&\Leftrightarrow [\ln P(s)]_0^t = [rs]_0^t \\
&\Leftrightarrow \ln P(t) - \ln P(0) = rt \\
&\Leftrightarrow \ln \frac{P(t)}{P(0)} = rt \\
&\Leftrightarrow P(t) = P(0)e^{rt},
\end{aligned}$$

a different expression for the Hotelling rule.