Stochastic Calculus - part 18

ISEG

The Black-Scholes model

- Black-Scholes model: 2 assets with dynamics

$$dB(t) = rB(t)\,dt, \tag{1}$$
$$dS_t = \alpha S_t dt + \sigma S_t d\overline{W}_t, \tag{2}$$

where r, α and σ are parameters.

- $B(t)$ represents the deterministic price of a riskless asset (a bond or a bank deposit).
- S_t is the (stochastic) price process of a risky asset (a stock or an index).
- $\overline{W}_t$ is a standard Brownian motion with respect to the original probability measure P.

The Black-Scholes model

- r: risk-free interest rate (or short rate of interest).
- α: mean rate of return of the risky asset
- σ: Volatility of the risky asset
- The solution of (2) is the geometric Brownian motion:

$$S_t = S_0 \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma \overline{W}_t\right).$$

Financial Derivatives

- Consider a contingent claim (a financial derivative), with payoff given by
$$\chi = \Phi\left(S\left(T\right)\right). \tag{3}$$
Its price process is represented by
$$\Pi\left(t\right), \quad t \in \left[0, T\right].$$

Portfolios

- Portfolio $\left(h^0(t), h^*(t)\right)$
- $h^0(t)$: number of bonds (or number of units of the riskless asset) at time t.
- $h^*(t)$: number of of shares of stock in the portfolio at time t.

Portfolios

- Value of the portfolio at time t:

$$V^h(t) = h^0(t) B_t + h^*(t) S_t.$$

- It is supposed that the portfolio is self-financed, that is,

$$dV_t^h = h^0(t) dB_t + h^*(t) dS_t.$$

- In integral form:

$$\begin{aligned} V_t &= V_0 + \int_0^t h^*(s) dS_s + \int_0^t h^0(s) dB_s \\ &= V_0 + \int_0^t \left(\alpha h^*(s) S_s + r h^0(s) B_s\right) ds + \sigma \int_0^t h^*(s) S_s d\overline{W}_s. \end{aligned} \tag{4}$$

Black-Scholes model

- Assume that the contingent claim (or financial derivative) has the payoff
$$\chi = \Phi\left(S\left(T\right)\right). \tag{5}$$
and it is replicated by the portfolio $h = \left(h^0\left(t\right), h^*\left(t\right)\right)$, that is, $V_T^h = \chi = \Phi\left(S\left(T\right)\right)$ a.s. Then, the unique price process that is compatible with the no-arbitrage principle is
$$\Pi\left(t\right) = V_t^h, \quad t \in \left[0, T\right]. \tag{6}$$
- Moreover, assume also that
$$\Pi\left(t\right) = V_t^h = F\left(t, S_t\right). \tag{7}$$
where F is a differentiable function of class $C^{1,2}$.

Black-Scholes model

- Applying It's formula to (7) and considering (2),
$$dF(t, S_t) = \left(\frac{\partial F}{\partial t}\left(t, S_t\right) + \alpha S_t \frac{\partial F}{\partial x}\left(t, S_t\right) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}\left(t, S_t\right)\right) dt + \left(\sigma S_t \frac{\partial F}{\partial x}\left(t, S_t\right)\right) d\overline{W}_t.$$

Black-Scholes model

That is,

$$F(t, S_t) = F(0, S_0) + \int_0^t \left(\frac{\partial F}{\partial t}(s, S_s) + AF(s, S_s) \right) ds$$
$$+ \int_0^t \left(\sigma S_s \frac{\partial F}{\partial x}(s, S_s) \right) d\overline{W}_s, \qquad (8)$$

where

$$Af(t, x) = \alpha x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x)$$

is the infinitesimal generator associated to the diffusion S_t.

Black-Scholes model

- Comparing (4) and (8), we have

$$\sigma h^*(s) S_s = \sigma S_s \frac{\partial F}{\partial x}(s, S_s),$$
$$\alpha h^*(s) S_s + r h^0(s) B_s = \frac{\partial F}{\partial t}(s, S_s) + AF(s, S_s).$$

- Therefore,

$$\frac{\partial F}{\partial x}(s, S_s) = h^*(s),$$
$$\frac{\partial F}{\partial t}(s, S_s) + r S_s \frac{\partial F}{\partial x}(s, S_s) + \frac{1}{2}\sigma^2 S_s^2 \frac{\partial^2 F}{\partial x^2}(s, S_s) - rF(s, S_s) = 0.$$

Black-Scholes model

Therefore, we have

- A portfolio h with value $V_t^h = F(t, S_t)$, composed of risky assets with price S_t and riskless assets of price B_t.
- Portfolio h replicates the contingent claim χ at each time t, and

$$\Pi(t) = V_t^h = F(t, S_t).$$

- In particular,

$$F(T, S_T) = \Phi(S(T)) = \text{Payoff}.$$

Black-Scholes model

- The portfolio should be continuously updated by acquiring (or selling) $h^*(t)$ shares of the risky asset and $h^0(t)$ units of the riskless asset, where

$$h^*(t) = \frac{\partial F}{\partial x}(t, S_t),$$
$$h^0(t) = \frac{V_t^h - h^*(t) S_t}{B_t} = \frac{F(t, S_t) - h^*(t) S_t}{B_t}.$$

- The derivative price function satisfies the PDE (Black-Scholes eq.)

$$\frac{\partial F}{\partial t}(t, S_t) + rS_t \frac{\partial F}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}(t, S_t) - rF(t, S_t) = 0.$$

Black-Scholes model

Theorem

(Black-Scholes eq.) Suppose that the market is specified by eqs. (1)-(2) and we want to price a derivative with payoff (3). Then, the only pricing function that is consistent with the no-arbitrage principle is the solution F of the following boundary value problem, defined in the domain $[0, T] \times \mathbb{R}^+$*:*

$$\frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \quad (9)$$
$$F(T,x) = \Phi(x).$$

Black-Scholes model

- The Black-Scholes equation may be solved analytically or with probabilistic methods.

Proposition

(Feynman-Kac formula) Let F be a solution of the boundary values problem

$$\frac{\partial F}{\partial t}(t,x)+\mu(t,x)\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)-rF(t,x)=0, \tag{10}$$
$$F(T,x)=\Phi(x).$$

Assume that $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is a process in L^2 (i.e. $E\int_0^t\left(\frac{\partial F}{\partial x}(s,X_s)\sigma(s,X_s)\right)^2 ds<\infty$). Then,

$$F(t,x)=e^{-r(T-t)}E_{t,x}\left[\Phi(X_T)\right],$$

where X satisfies

$$dX_s=\mu(s,X_s)\,ds+\sigma(s,X_s)\,dB_s,$$
$$X_t=x.$$

Black-Scholes model

- Applying the Feynman-Kac formula from the previous proposition to the eq. (9), we obtain:

$$F(t,x)=e^{-r(T-t)}E_{t,x}\left[\Phi(X_T)\right], \tag{11}$$

where X is a stochastic process with dynamics:

$$dX_s=rX_s ds+\sigma X_s d\overline{W}_s, \tag{12}$$
$$X_t=x.$$

Black-Scholes model

- Note that the process X is not the same as the process S, as the drift of X is rX and not αX.
- idea: change from process X to process S, using the Girsanov Theorem.

- Denote by P the original probability measure ("objective" or "real" probability measure). The P-dynamics of the process S is given in (2).
- Note that (2) is equivalent to

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t \left(\frac{\alpha - r}{\sigma} dt + d\overline{W}_t \right) \\
&= rS_t dt + \sigma S_t d \underbrace{\left(\frac{\alpha - r}{\sigma} t + \overline{W}_t \right)}_{W_t}.
\end{aligned}
$$

- By the Girsanov Theorem, there exists a probability measure Q such that, in the probability space $(\Omega, \mathcal{F}_T, Q)$, the process

$$
W_t := \frac{\alpha - r}{\sigma} t + \overline{W}_t
$$

is a Brownian motion, and S has the Q-dynamics:

$$
dS_t = rS_t dt + \sigma S_t dW_t. \tag{13}
$$

- consider the following notation: E denotes the expected value with respect to the original measure P, while E^Q denotes the expected value with respect to the new probability measure Q (that comes from the application of the Girsanov theorem). Also, let $\overline{W}_t$ denote the original Brownian motion (under the measure P) and W_t denote the Brownian motion under the measure Q.
- Getting back to (11) and (12), and taking into account that under the measure Q the equations (12) and (13) are the same, we may represent the solution of the Black-Scholes equation by

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right],$$

where the dynamics of S under the measure Q is

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

We may finally state the theorem that provides us a pricing formula for the contingent claim in terms of the new measure Q.

Theorem

The price (absent of arbitrage) of the contingent claim $\Phi(S_T)$ is given by the formula

$$F\left(t, S_t\right) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S_T)\right], \quad (14)$$

where the dynamics of S under the measure Q is

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

- In the Black-Scholes, the diffusion coefficient σ may depend on t and S - be a function $\sigma(t, S_t)$ - and in this case, the calculations needed would be analogous to the ones we have done.
- The measure Q is called equivalent martingale measure. The reason for this has to do with the fact that the discounted process

$$\widetilde{S}_t := \frac{S_t}{B_t}$$

is a Q-martingale (martingale under the measure Q). In fact,

$$\begin{aligned}\widetilde{S}_t &= \frac{S_t}{B_t} = e^{-rt} S_t = e^{-rt} S_0 \exp\left(\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma \overline{W}_t\right) \\ &= S_0 \exp\left(-\frac{1}{2}\sigma^2 t + \sigma W_t\right)\end{aligned}$$

is a martingale.