Stochastic Calculus - part 19

ISEG

Risk neutral Valuation formula

Theorem

The price (considering the no arbitrage principle) of the contingent claim $\Phi(S_T)$ is given by $F(t, S_t)$ where F is given by the risk-neutral valuation formula

$$F(t, S_t) = e^{-r(T-t)} E^{Q}_{t,s}\left[\Phi(S_T)\right], \quad (1)$$

where the dynamics of S under the Q measure is

$$dS_u = rS_u du + \sigma S_u dW_u, \quad (2)$$

$$S_t = s. \quad (3)$$

Risk neutral valuation formula

- Calculating explicitly the price of the derivative, we have

$$\begin{aligned} & e^{r(T-t)} F(t,s) = E^{Q}_{t,s}\left[\Phi(S_T)\right] \\ & = E^{Q}_{t,s}\left[\Phi\left(s\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\left(W_T-W_t\right)\right)\right)\right] \\ & = E^{Q}\left[\Phi\left(se^{Z}\right)\right], \end{aligned}$$

where $Z = \left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\left(W_T-W_t\right) \sim N\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$.

- Therefore

$$F(t,s) = e^{-r(T-t)}\int_{-\infty}^{+\infty}\Phi\left(se^{y}\right)f(y)\,dy, \quad (4)$$

where f is the p.d.f. of the Gaussian r.v. Z.

Risk neutral valuation formula

- The integral formula (4), for a given function Φ, should be, in the general case, computed using numerical methods.
- However, there are some particular cases where (4) may be obtained analytically. For example, for a European "call" option with payoff

$$\Phi(x) = (x-K)^{+} = \max(x-K,0),$$

we have

$$\begin{aligned} F(t,s) &= e^{-r(T-t)}\int_{-\infty}^{+\infty}\max\left(se^{y}-K,0\right)f(y)\,dy \\ &= e^{-r(T-t)}\int_{\ln(K/s)}^{+\infty}\left(se^{y}-K\right)f(y)\,dy \\ &= e^{-r(T-t)}\left(s\int_{\ln(K/s)}^{+\infty}e^{y}f(y)\,dy - K\int_{\ln(K/s)}^{+\infty}f(y)\,dy\right) \end{aligned} \quad (5)$$

Risk neutral valuation formula

- Auxiliary computations

$$\int_{\ln(K/s)}^{+\infty} e^y f(y)\, dy =$$

$$= \int_{\ln(K/s)}^{+\infty} \frac{\exp\left(y - \frac{\left(y-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right)}{\sigma\sqrt{2\pi(T-t)}} dy$$

$$= \int_{\ln(K/s)}^{+\infty} \frac{\exp\left(\frac{2\sigma^2(T-t)y-\left(y-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right)}{\sigma\sqrt{2\pi(T-t)}} dy$$

$$= e^{r(T-t)} \int_{\ln(K/s)}^{+\infty} \frac{\exp\left(-\frac{\left(y-\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right)}{\sigma\sqrt{2\pi(T-t)}} dy$$

Since $\frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left(-\frac{\left(y-\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right)$ is the density function of a random variable Z^* with distribution $N\left(\left(r+\frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$.

Risk neutral valuation formula

- Auxiliary computations

$$\int_{\ln(K/s)}^{+\infty} e^{y} f(y)\, dy = e^{r(T-t)} Q\left(Z^* \geq \ln(K/s)\right)$$
$$= e^{r(T-t)} Q\left(\overline{Z} \geq \frac{\ln(K/s) - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right)$$
$$= e^{r(T-t)} Q\left(\overline{Z} \leq \frac{\ln(s/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right)$$
$$= e^{r(T-t)} N\left[d_1(t,s)\right],$$

where $N[x]$ is the cumulative distribution function of the distribution $N(0,1)$ and

$$d_1(t,s) = \frac{\ln(s/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}.$$

Risk neutral valuation formula

- Auxiliary computations:

$$\int_{\ln(K/s)}^{+\infty} f(y)\,dy = Q(Z \geq \ln(K/s))$$
$$= Q\left(\overline{Z} \geq \frac{\ln(K/s) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right)$$
$$= Q\left(\overline{Z} \leq \frac{\ln(s/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}\right) = N[d_2(t,s)],$$

where $N[x]$ is the cumulative distribution function of the distribution $N(0,1)$ and

$$d_2(t,s) = \frac{\ln(s/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

Risk neutral valuation formula

- Using the auxiliary computations in (5), we get

$$F(t,s) = e^{-r(T-t)}\left(se^{r(T-t)}N[d_1(t,s)] - KN[d_2(t,s)]\right)$$
$$= sN[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)]$$

- Black-Scholes formula:

$$F(t,s) = sN[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)].$$

Exercise

Deduce the Black-Scholes Formula form (1) for an European put option (with payoff $\Phi(S_T) = \max(K - S_T, 0)$).