2.2 – SHORT RATE MODELS

- Interest Rate Trees
- Single-Factor short rate models
- Multi-Factor short rate models

- How can we infer a model for interest rate dynamics r_u and for all future zero coupon curves $T \mapsto L(u,T)$, $u > t$, only from knowledge of today's zero-coupon curve $T \mapsto L(t,T)$, with $t =$ today?

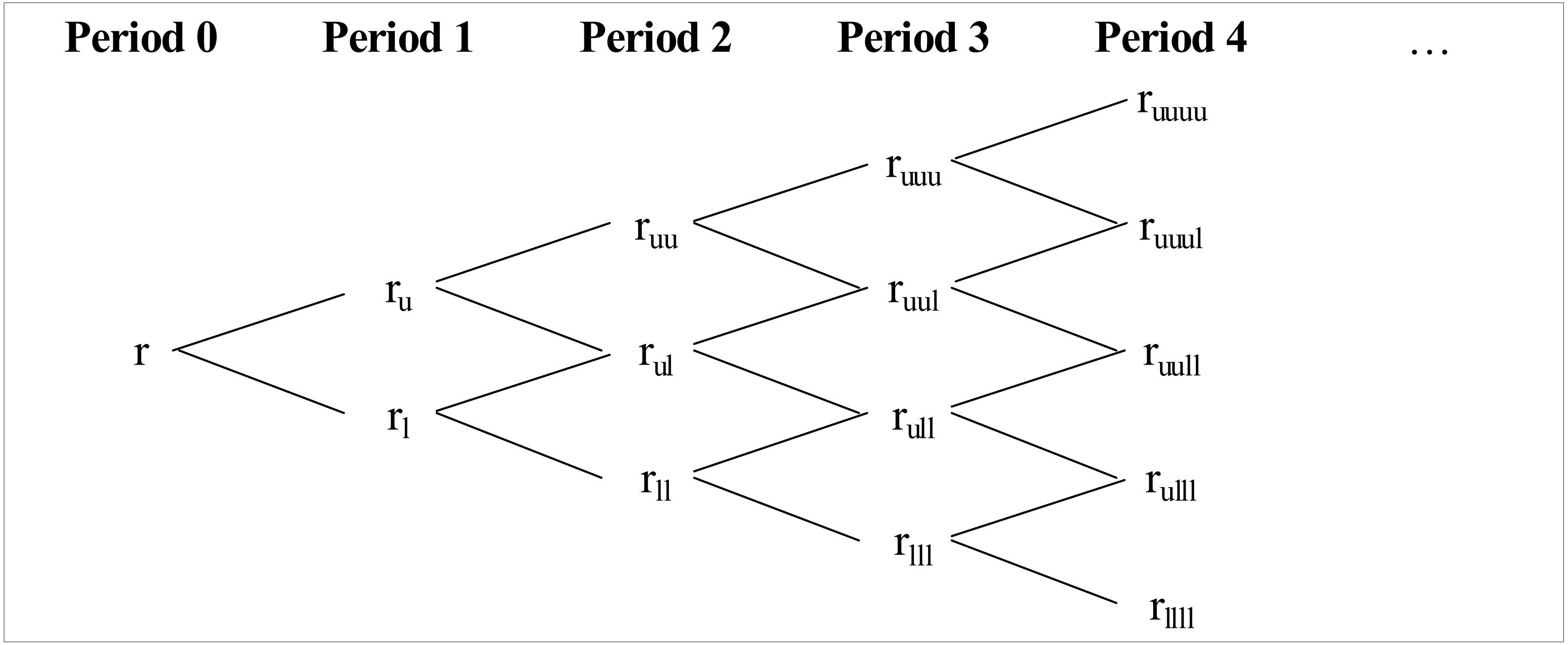
We may do so by assuming a certain structure for the risk-neutral dynamics of r.

2.2.1 – INTEREST RATE TREES

- General binomial model
  - Given the current level of short-term rate r, the next-period short rate can take only two possible values: an upper value r_u and a lower value r_l, with equal probability 0.5
  - In period 2, the short-term interest rate can take on four possible values: r_{uu}, r_{ul}, r_{lu}, r_{ll}
  - More generally, in period *n*, the short-term interest rate can take on 2^n values => very time-consuming and computationally inefficient

- Recombining trees
  - Means that an upward-downward sequence leads to the same result as a downward-upward sequence
  - For example, $r_{ul} = r_{lu}$
  - Only (n+1) different values at period n

INTEREST RATE TREE - Recombining



INTEREST RATE TREE – analytical

- We may write down the binomial process as

$$\Delta r_t \equiv r_{t+1} - r_t = \sigma\varepsilon_t$$

where ε_t are independent variables taking on values (+1,-1) with probability (1/2,1/2)

- Problem: rates can take on negative values with positive probability

- Fix that problem by working with logs

$$\Delta \ln r_t \equiv \ln r_{t+1} - \ln r_t = \sigma\varepsilon_t$$

$$\Rightarrow r_{t+1} = r_t \times \exp\left(\sigma\varepsilon_t\right) = r_t \times \begin{pmatrix} u = \exp\left(\sigma\right) \\ d = \exp\left(-\sigma\right) \end{pmatrix}$$

with probability (1/2,1/2)

- More general models (could be written on log rates)

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu\left(t,\Delta t,r_t\right) + \sigma\left(t,\Delta t,r_t\right)\varepsilon_t$$

- Specific case

$$\Delta r_t \equiv r_{t+\Delta t} - r_t = \mu\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon_t$$

- Continuous-time limit (Merton (1973))

$$dr_t \equiv r_{t+dt} - r_t = \mu dt + \sigma dW_t$$

INTEREST RATE TREE – calibration

- Calibration of the model is performed so as to make model consistent with the current term structure
- We have at date 0

$$\Delta \ln r_0 \equiv \ln r_{\Delta t} - \ln r_0 = \mu \Delta t + \sigma \varepsilon_0 \sqrt{\Delta t}$$

$$\Rightarrow \ln r_u - \ln r_l = 2\sigma \sqrt{\Delta t} \quad \text{or} \quad r_u = r_l \exp\left(2\sigma \sqrt{\Delta t}\right)$$

- We take as given an estimate for σ, the current yield curve y_t, and we iteratively find the values r_u, r_l, r_{uu}, r_{ul}, r_{lu}, r_{ll}, etc., consistent with the input data

- Consider a 2 period tree with $\Delta t = 1$
- The price one year from now of a 2-year Treasury bond (at the par value) can take two values: a value P_u associated with r_u, and a value P_l, associated with r_l

$$P_u = \frac{100 + y_2}{1 + r_u} \quad \text{and} \quad P_d = \frac{100 + y_2}{1 + r_l}$$

NPV of future cash-flows (redemption and coupon)

- Then, taking expectations at time 0, we find an equation that can be solved for r_u and r_l , given that

$$r_u = r_l \exp\left(2\sigma\sqrt{\Delta t}\right)$$

$$100 = \frac{1}{2}\left(\frac{\frac{100 + y_2}{1 + r_l \exp\left(2\sigma\right)} + y_2}{1 + y_1} + \frac{\frac{100 + y_2}{1 + r_l} + y_2}{1 + y_1} \right)$$

1st year coupon

INTEREST RATE TREE – example

- Consider a 2-year Treasury bond with a coupon rate y_2 = 4.30% and being the 1-year rate y_1 = 4%.
- We want to calibrate a binomial interest rate tree, assuming a volatility of 1% for the one-year interest rate.

.02=2* $\sigma = 2 * 0{,}01$

$$100 = \frac{1}{2}\left(\frac{\dfrac{100+4.3}{1+r_l \exp(.02)} + 4.3}{1+4\%} + \frac{\dfrac{100+4.3}{1+r_l} + 4.3}{1+4\%} \right)$$

$$\Rightarrow \begin{cases} r_l = 4.57\% \\ r_u = 4.66\% \end{cases}$$

2.2.2 – CT Single Factor Models

- General expression for a single-factor continuous-time model

$$dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t)\,dW_t$$

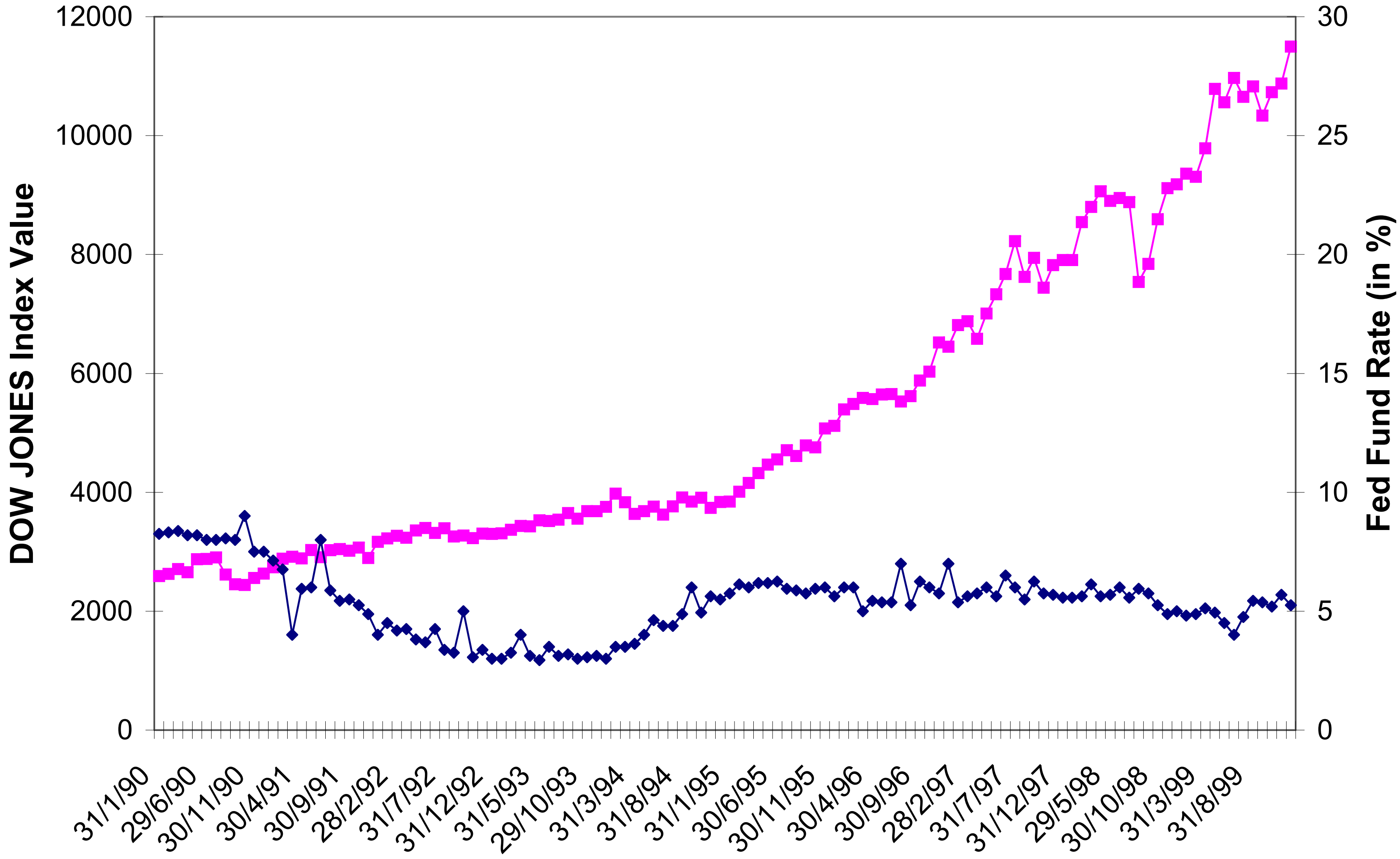
- The term *W* denotes a Brownian motion - process with independent normally distributed increments:
  - *dW* represents the instantaneous change.
  - It is stochastic (uncertain)
  - It behaves as a normal distribution with zero mean and variance *dt*
  - It can be thought of as

$$dW_t = \varepsilon_t \sqrt{dt}$$

WHAT IS A GOOD MODEL?

- A good model is a model that is consistent with reality
- Stylized facts about the dynamics of the term structure
  - Fact 1: (nominal) interest rates are positive
  - Fact 2: interest rates are mean-reverting
  - Fact 3: interest rates with different maturities are imperfectly correlated
  - Fact 4: the volatility of interest rates evolves (randomly) in time
- A good model should also be
  - Tractable
  - Parsimonious

Empirical Facts 1, 2 and 4



Empirical Fact 3

	1M	3M	6M	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
1M	1												
3M	0.992	1											
6M	0.775	0.775	1										
1Y	0.354	0.3	0.637	1									
2Y	0.214	0.165	0.42	0.901	1								
3Y	0.278	0.246	0.484	0.79	0.946	1							
4Y	0.26	0.225	0.444	0.754	0.913	0.983	1						
5Y	0.224	0.179	0.381	0.737	0.879	0.935	0.981	1					
6Y	0.216	0.168	0.352	0.704	0.837	0.892	0.953	0.991	1				
7Y	0.228	0.182	0.35	0.661	0.792	0.859	0.924	0.969	0.991	1			
8Y	0.241	0.199	0.351	0.614	0.745	0.826	0.892	0.936	0.968	0.992	1		
9Y	0.238	0.198	0.339	0.58	0.712	0.798	0.866	0.913	0.95	0.981	0.996	1	
10Y	0.202	0.158	0.296	0.576	0.705	0.779	0.856	0.915	0.952	0.976	0.985	0.99	1

Daily changes in French swap markets in 1998

POPULAR ENDOGENOUS SHORT RATE CT MODELS

- Dynamics of the short-term rate (here represented by x_t) under the risk neutral world probability measure:

1. **Vasicek (1977):**

$$dx_t = k(\theta - x_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma).$$

Constant volatility models

2. **Cox-Ingersoll-Ross (CIR, 1985):**

Stochastic volatility models

$$dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t, \quad \alpha = (k, \theta, \sigma), \quad 2k\theta > \sigma^2 .$$

3. **Dothan / Rendleman and Bartter:**

$$dx_t = ax_t dt + \sigma x_t dW_t, \quad (x_t = x_0\, e^{(a - \frac{1}{2}\sigma^2)t + \sigma W_t}, \quad \alpha = (a, \sigma)).$$

4. **Exponential Vasicek:**

$$x_t = \exp(z_t), \quad dz_t = k(\theta - z_t)dt + \sigma dW_t, \quad \alpha = (k, \theta, \sigma).$$

VASICEK MODEL

- The Vasicek model has some peculiarities that make it attractive:

- Linear equation
- Gaussian disturbances
- Mean reverting – expected value of the short rate tends to a constant value θ, with velocity given by k.

However, this model features also some drawbacks.

- Rates can assume negative values with positive probability.

- Gaussian distributions for the rates are not compatible with the market implied distributions.

Cox-Ingersoll and Ross Model

$$dx(t) = k[\theta - x(t)]dt + \sigma \boxed{\sqrt{x(t)}} dW(t), \quad r_t = x_t$$

- The model is mean reverting: The expected value of the short rate tends to a constant value θ with velocity depending on k as time grows towards infinity, while its variance does not explode.

- This model maintains a certain degree of analytical tractability, but is **less tractable** than Vasicek, especially as

On the other hand,

- CIR is usually closer to market implied distributions of rates than Vasicek.

WHY ARE THESE MODELS CALLED ENDOGENOUS?

- Because they can be computed as a function of the parameters of the dynamics of the short rate itself.

- For example, in Vasicek and CIR, given k, θ, σ and $r(t)$, once the function $T \mapsto P(t, T; k, \theta, \sigma, r(t))$ is known, we know the whole interest-rate curve at time t. At $t = 0$ (initial time), the interest rate curve is an **output** of the model, rather than an input, depending on k, θ, σ, r_0 in the dynamics.

POPULAR EXOGENOUS SHORT RATE CT MODELS

Exogenous short-rate models are built by suitably modifying the above endogenous models. The basic strategy that is used to transform an endogenous model into an exogenous model is the inclusion of "time-varying" parameters.

Dynamics of $r_t = x_t$ under the risk–neutral measure:

1. **Ho-Lee:**

$$dx_t = \theta(t)\ dt + \sigma\ dW_t.$$

2. **Hull-White (Extended Vasicek):**

$$dx_t = k(\theta(t) - \ x_t)dt + \sigma dW_t.$$

3. **Hull-White (Extended CIR):**

$$dx_t = k(\theta(t) - \ x_t)dt + \sigma\ \sqrt{x_t}\ dW_t\ .$$

4. **Black-Derman-Toy (Extended Dothan):**

$$x_t = x_0 \, e^{u(t)+\sigma(t)W_t}$$

5. **Black-Karasinski (Extended exponential Vasicek):**

$$x_t = \exp(z_t), \quad dz_t = k\left[\theta(t) - z_t\right]dt + \sigma dW_t.$$

6. **CIR++ (Shifted CIR model, Brigo & Mercurio (2000)):**

$$r_t = x_t + \phi(t;\alpha), \quad dx_t = k(\theta - x_t)dt + \sigma\sqrt{x_t}dW_t$$

2.2.3 – CT MULTI FACTOR MODELS

MOST POPULAR MODELS

- **Fong and Vasicek (1991) model**
  - Fong and Vasicek (1991) take the short rate and its volatility as two state variables
  - Variance of the short-rate changes is a key element in the pricing of fixed-income securities, in particular interest rates derivatives

- **Longstaff and Schwartz (1992) model**
  - Longstaff and Schwartz (1992) use the same two state variables, but with a different specification
  - Allows them to get closed-form solution for the price of a discount bond and a call option on a discount bond

Chen (1996) and Balduzzi et al. (1996) models

- Chen (1996) and Balduzzi et al. (1996) suggest the use of a three-factor model by adding the short-term average of the short rate.

- These three state variables can be assimilated to the three factors which can be empirically obtained through a principal components analysis of the term structure dynamics.

2.2.4 – MODELING THE TERM STRUCTURE: AFFINE MODELS

- Fundamental asset pricing concept - The pricing of any financial asset is based on a very intuitive result - the price corresponds to the present value of the future asset pay-off:

(1) $$P_t = E_t[P_{t+1}M_{t+1}]$$

being *Pt* the price of a financial asset providing nominal cash-flows and *Mt*+1 the nominal stochastic discount factor (sdf) or pricing kernel, as it is the determining variable of *Pt* . In fact, solving equation (1) forward, the asset price may be written solely as a function of the pricing kernel, as:

(2) $$P_t = E_t[M_{t+1} \ldots M_{t+n}]$$

- Asset prices and returns are related to their risk, i.e., to the asset capacity of offering higher cash-flows when they are more needed and valued.
- Actually, the more an asset helps to smooth income fluctuations, the less risky it is and the higher will be its demand for ensuring against "bad times".
- Considering that

$$E(XY) = E(X)E(Y) + COV(X,Y)$$

- Equation (1) may be written as:

(3) $$P_t = E_t[P_{t+1}]E_t[M_{t+1}] + Cov_t[P_{t+1}, M_{t+1}]$$

- When the asset is riskless, its pay-off in *t*+1 is known in *t* with certainty. Thus, it may be considered as a constant in *t*, which implies, from (1):

(4) $$\frac{P_t}{P_{t+1}} = E_t[M_{t+1}]$$

- As the LHS of (4) is the inverse of the risk-free asset's gross return, denoted by $1+i_{t+1}^{f}$, replacing in equation (3) $E_t[M_{t+1}]$ by $1/1+i_{t+1}^{f}$, it is obtained:

(5) $$P_t = E_t[P_{t+1}]\frac{1}{1+i_{t+1}^{f}} + Cov_t[P_{t+1}, M_{t+1}]$$

- This result shows that the asset price is the discounted expected value of its future pay-off or price, adjusted by the covariance of its return with the sdf.
- As it will become clear later, this covariance consists in a risk factor and it is positive for assets that pay higher returns when they are more needed.
- The same result may be obtained for interest rates, instead of prices. Actually, dividing both sides of equation (1) by P_t, one gets:

(6) $$1 = E_t\left[(1 + i_{t+1})M_{t+1}\right]$$

- Applying the well-known statistical result

$$E(XY) = E(X)E(Y) + COV(X,Y)$$

- to (6) it is obtained

$$E_t(1+i_{t+1})\cdot E_t(M_{t+1}) + Cov(i_{t+1}, M_{t+1}) = 1 \Leftrightarrow E_t(1+i_{t+1}) = \frac{[1 - Cov(i_{t+1}, M_{t+1})]}{E_t(M_{t+1})}$$

- Following equation (4) we obtain:

$$E_t(1+i_{t+1}) = \frac{1}{E_t(M_{t+1})} - \frac{Cov(i_{t+1}, M_{t+1})}{E_t(M_{t+1})} \Leftrightarrow E_t(1+i_{t+1}) = (1+i^f_{t+1}) - \frac{Cov(i_{t+1}, M_{t+1})}{E_t(M_{t+1})}$$

- Therefore, we get:

(7) $$E_t[i_{t+1}] = i^f_{t+1} - \frac{Cov_t[M_{t+1}, i_{t+1}]}{E_t[M_{t+1}]}$$

The interest rate of an asset results from the risk-free rate, adjusted by a risk factor => the lower the covariance, the higher the risk and the interest rate.

- With some additional self-explanatory algebra, the following result is obtained:

(8) $$E_t[i_{t+1}] = i^f_{t+1} + \frac{Cov_t[M_{t+1}, i_{t+1}]}{Var_t[M_{t+1}]} \cdot \left(-\frac{Var_t[M_{t+1}]}{E_t[M_{t+1}]} \right) = i^f_{t+1} + \beta_{i_{t+1},M_{t+1}} \lambda.$$

- In equation (8), $\beta_{i_{t+1},M_{t+1}}$ is the coefficient of a regression of i_{t+1} on M_{t+1}.
- Therefore, it measures the correlation between the asset's return and the stochastic discount factor (sdf) or the quantity of risk.

- Market price of risk: $$\lambda = -\frac{Var_t[M_{t+1}]}{E_t[M_{t+1}]}$$

- From equation (7), denoting $\rho_{M_{t+1},i_{t+1}}$ by the correlation coefficient between the sdf and the asset's rate of return and $\sigma_{M_{t+1}}$ and $\sigma_{i_{t+1}}$, the excess return of any asset over the risk-free asset is:

(9) $$\Lambda_t = E_t[i_{t+1}] - i^f_{t+1} = -\rho_{M_{t+1},i_{t+1}} \frac{\sigma_{M_{t+1}} \sigma_{i_{t+1}}}{E_t[M_{t+1}]}$$

- Equations (7) and (9) illustrate a basic result in finance theory: the excess return of any asset over the risk-free asset depends on the covariance of its rate of return with the sdf => an asset with pay-off negatively correlated to the sdf is riskier.

- The mean-variance frontier will correspond to the limiting values of equation (9) => expected values and standard-deviations must lie in the interval $\left[-\frac{\sigma_{M_{t+1}}\sigma_{i_{t+1}}}{E_t[M_{t+1}]}, \frac{\sigma_{M_{t+1}}\sigma_{i_{t+1}}}{E_t[M_{t+1}]}\right]$.

mean-variance region

minimum risk (frontier): $\rho_{M_{t+1},i_{t+1}} = 1$

- As on the frontier all asset returns are perfectly correlated with the sdf, all asset returns are also perfectly correlated with each other => it is possible to define the return of any asset as a linear combination of the returns of any 2 other assets - market or wealth portfolio and the risk-free asset:

(10) $E_t[i_{t+1}] = \beta_{i_{t+1},i^W_{t+1}} E[i^W_{t+1}] + (1-\beta_{i_{t+1},i^W_{t+1}}) i^f_{t+1} = i^f_{t+1} + \beta_{i_{t+1},i^W_{t+1}} (E[i^W_{t+1}] - i^f_{t+1})$

i^W_{t+1} - Rate of return of market portfolio

CAPM

(8) $$E_t\left[i_{t+1}\right]=i_{t+1}^f+\frac{Cov_t\left[M_{t+1},i_{t+1}\right]}{Var_t\left[M_{t+1}\right]}\cdot\left(-\frac{Var_t\left[M_{t+1}\right]}{E_t\left[M_{t+1}\right]}\right)=i_{t+1}^f+\beta_{i_{t+1},M_{t+1}}\lambda.$$

(10) $$E_t\left[i_{t+1}\right]=\beta_{i_{t+1},i_{t+1}^W}E\left[i_{t+1}^W\right]+\left(1-\beta_{i_{t+1},i_{t+1}^W}\right)i_{t+1}^f=i_{t+1}^f+\beta_{i_{t+1},i_{t+1}^W}\left(E\left[i_{t+1}^W\right]-i_{t+1}^f\right)$$

- (8) + (10) => CAPM assumes the sdf as a function of the gross rate of return of the wealth market portfolio, while the market price of risk is the spread between the expected market portfolio return and the risk-free asset return.

- **CCAPM:** an asset will pay a higher return or is riskier when the covariance of its return with the marginal utility of consumption is lower, i.e. when consumption is higher.

- Affine models: log-linear relationship between asset prices and the sdf, on one side, and the factors or state variables, on the other side.
- These models were originally developed by Duffie and Kan (1996), for the term structure of interest rates.
- Equation (1) in logs:

(11) $$p_t = \log\left(E_t\left[P_{t+1}M_{t+1}\right]\right)$$

- Assuming joint log-normality of asset prices and discount factor => if *log X ~ N*(μ,σ^2) then *log E*(*X*) = $\mu + \sigma^2/2$ => basic equation considered in the affine models:

(12) $$p_t = E_t\left[m_{t+1} + p_{t+1}\right] + 0.5 \cdot Var_t\left[m_{t+1} + p_{t+1}\right]$$

- DK models: multifactor affine models of the term structure, where the pricing kernel is a linear function of several factors

$$z_t^T = \left(z_{1,t} \ldots z_{k,t}\right)$$

- <u>DK models advantages</u>:

(i) Accommodate the most important term structure models, from Vasicek (1977) and CIR one-factor models to multi-factor models.

(ii) Allow the estimation of the term structure simultaneously on a cross-section and time-series basis.

(iii) Provide a way of computing and estimating simple closed-form expressions for the spot, forward, volatility and term premium curves.

- Discount factors:

(13) $$-m_{t+1} = \xi + \gamma^T z_t + \lambda^T V(z_t)^{1/2} \varepsilon_{t+1}$$

$V(Z_t)$ - variance matrix of the random shocks on the sdf, defined as a diagonal matrix with elements $v_i(z_t) = \alpha_i + \beta_i^T z_t$

ε_t - independent shocks $\varepsilon_t \sim N(0, I)$

λ^T **-** market prices of risks, as they govern the covariance between the stochastic discount factor and the yield curve factors.

- Higher λs ⇔ higher covariance between the discount factor and the asset return ⇔ lower expected rate of returns or the less risky the asset is.

- Another way to write the pricing kernel (from (13)):

(14)

$$-m_{t+1} = \xi + \gamma_1 z_{1t} + \gamma_2 z_{2t} + \ldots + \gamma_k z_{kt} + \lambda_1 \sigma_{1t} \varepsilon_{1,t+1} + \lambda_2 \sigma_{2t} \varepsilon_{2,t+1} + \ldots \lambda_1 \sigma_{kt} \varepsilon_{k,t+1}$$

- The k-dimensional vector of factors z_t is defined as follows:

(15) $$z_{t+1} = (I - \Phi)\theta + \Phi z_t + V(z_t)^{1/2} \varepsilon_{t+1}$$

Φ – has positive diagonal elements, ensuring that the factors are stationary;
θ - long-run mean of the factors.

- From (15), we have the factors as follows:

(16) $z_{i,t+1} = (1-\phi_i)\theta_i + \phi_i z_{i,t} + \sigma_{i,t}\varepsilon_{i,t+1}$, where $\sigma_{it} = \sqrt{\alpha_i + \beta_{i1} z_{1t} + \beta_{i2} z_{2t} + \ldots + \beta_{ik} z_{kt}}$

- <u>Asset prices are also log-linear functions of the factors</u>.

(17) $$-p_{n,t} = A_n + B_n^{\ T} z_t$$

n - term to maturity

A_n and B_n - vectors of parameters to be estimated.

B_n - factor loadings (impact of a random shock on the factors over the log of asset prices).

- In term structure models, the identification of the parameters is easier assuming that the term structure is modelled using zero-coupon bonds paying 1 monetary unit => the log of the maturing bond price = 0 => (from (17)) $A_0 = B_0 = 0$
- It is possible to show that the factor loading can be obtained recursively:

(18)
$$A_n = A_{n-1} + \xi + B_{n-1}{}^T (I - \Phi)\theta - \frac{1}{2}(\lambda + B_{n-1})^T \alpha(\lambda + B_{n-1})$$
$$B_n{}^T = \gamma^T + B_{n-1}{}^T \Phi - \frac{1}{2}(\lambda + B_{n-1})^T \beta^T z_t (\lambda + B_{n-1})$$

- Considering that the continuously compounded yield is

(19) $$y_{n,t} = -\frac{\log P_{n,t}}{n}$$

- From (17) the yield curve is defined as:

(20) $$y_{n,t} = \frac{1}{n}\left(A_n + B_n{}^T z_t\right)$$

- From equations (18) and (20), as well as the normalisation $A_0 = B_0 = 0$, it is obtained the short-term rate:

(21) $$y_{1,t} = \xi - \frac{1}{2}\lambda^T \alpha \lambda + \left[\gamma^T - \frac{1}{2}\lambda^T \beta^T \lambda\right] z_t$$

- Correspondingly, using the definition of the factors in (15) and solving z_{t+n} backwards, the expected value of the short rate is:

(22) $$\begin{aligned} E_t\left(y_{1,t+n}\right) &= E_t\left(\xi - \frac{1}{2}\lambda^T \alpha \lambda + \left[\gamma^T - \frac{1}{2}\lambda^T \beta^T \lambda\right] z_{t+n}\right) \\ &= \xi - \frac{1}{2}\lambda^T \alpha \lambda + \left[\gamma^T - \frac{1}{2}\lambda^T \beta^T \lambda\right] E_t\left(z_{t+n}\right) \\ &= \xi - \frac{1}{2}\lambda^T \alpha \lambda + \left[\gamma^T - \frac{1}{2}\lambda^T \beta^T \lambda\right]\left[\left(I - \Phi^n\right)\theta + \Phi^n z_t\right] \end{aligned}$$

- Variance matrix in the specification of the factors (from (15) and (20)) => volatility of the yields:

(23)
$$Var_t\left(y_{n,t+1}\right)=\frac{1}{n^2}B_n{}^T V(z_t)B_n$$

- Instantaneous or one-period forward rate = log of the inverse of the gross return =>

$$f_{n,t}=p_{n,t}-p_{n+1,t}$$

- Instantaneous or one-period forward curve:

(24)
$$\begin{aligned}f_{n,t}&=\left(A_{n+1}+B_{n+1}{}^T z_t\right)-\left(A_n+B_n{}^T z_t\right)=\left(A_{n+1}-A_n\right)+\left(B_{n+1}{}^T-B_n{}^T\right)z_t=\\&=\left[\xi+B_n{}^T(I-\Phi)\theta-\frac{1}{2}\sum_{i=1}^{k}\left(\lambda_i+B_{i,n}\right)^2\alpha_i\right]+\left[\gamma^T+B_n{}^T(\Phi-I)-\frac{1}{2}\sum_{i=1}^{k}\left(\lambda_i+B_{i,n}\right)^2\beta_i^T\right]z_t\end{aligned}$$

- Term premium - one-period log excess return of the *n*-period bond over the short-rate:

(from (15) for the factor definition, (17) for the current and the one-period ahead bond prices and for the independent term in the price equation, (18) for the recursive restrictions on the factor loadings and (21) for the short-term interest rate)

(25)

$$\Lambda_{n,t} = E_t p_{n,t+1} - p_{n+1,t} - y_{1,t}$$
$$= -\sum_{i=1}^{k}\left[\lambda_i B_{i,n} + \frac{B_{i,n}^{\;2}}{2}\right]\alpha_i - \sum_{i=1}^{k}\left(\lambda_i B_{i,n} + \frac{B_{i,n}^{\;2}}{2}\right)\beta_i^T z_t$$

- The term premium can alternatively be calculated from the basic pricing equation (12):

(26) $$E_t p_{n,t+1} - p_{n+1,t} = -E_t m_{t+1} - Var_t i\left(i_{n,t+1}\right)/2 - Var_t\left(m_{t+1}\right)/2 - COV\left(i_{n,t+1}, m_{t+1}\right)$$

- **(12) +** $p_{ot} = 0$ **=>** short-term bond price:

(27) $$p_{1,t} = E_t\left[m_{t+1}\right] + \tfrac{1}{2}\cdot Var_t\left[m_{t+1}\right]$$

- **Term premium:**

(25) $$\Lambda_{n,t} = -COV_t(i_{n,t+1}, m_{t+1}) - Var_t(i_{n,t+1})/2$$

Short-term interest rate is obtained from (19) and (27).

Risk premium determined by the covar. of the asset's rate of return with the stochastic discount factor => the lower the covar., the higher the risk premium is.

(26) $$\Lambda_{n,t} = B_n^T COV(z_{t+1}, m_{t+1}) - B_n^T Var_t(z_{t+1}) B_n / 2$$

As from (17)

$$i_{n,t+1} = p_{n,t+1} - p_{n+1,t} = -A_n - B_n^T z_{t+1} + A_{n+1} + B_{n+1}^T z_t$$

$\Rightarrow$ the Cov in (25) is $-B_n^T COV_t(z_{t+1}, m_{t+1})$ while the Var of the factors is

$$B_n^T Var_t(z_{t+1}) B_n$$

- From (13) and (15):

$$\Lambda_{n,t} = -\lambda^T V(z_t) B_n - \frac{B_n^T V(z_t) B_n}{2}$$

at least one of the market prices of risk must be negative in order to have a positive term premium.

- One-factor models were the first step in modelling the term structure of interest rates.
- These models are grounded on the estimation of bond yields as functions of the short-term interest rate.
- Vasicek (1977) presented the whole term structure as a function of a single factor, the short-term interest rate, whose volatility was assumed to be constant.
- The Cox *et al*. (1985a) model added the stochastic volatility feature to the Vasicek model, avoiding interest rates to go negative, as in the Vasicek model. Thus, it corresponds to an analogous particular case of the DK model, with $\alpha_i = 0$ nd $\beta_i = \sigma_i^2$.

- Affine models may be classified according to the number of factors considered or to volatility properties.
- According to Litterman and Scheinkman (1991), the pronounced hump-shape of the US yield curve => 3 factors are required to explain the shifts in the whole term structure of interest rates.
- These factors are usually identified as the level, the slope and the curvature, being the level often responsible for the most important part of interest rate variation.
- Given the stochastic properties of interest rates volatility, Gaussian or constant volatility models are often rejected. Besides, these models impose constant volatility and one-period term premium curves (non-pure version of expectations theory).

- Additionally, the forward rate in these models exhibits some shortcomings. In fact, under constant volatility, the forward rate may be written as:

$$f_{n,t} = \xi + B_n^T (I - \Phi)\theta - \frac{1}{2}\sum_{i=1}^{k} (\lambda_i + B_{i,n})^2 \alpha_i + \left[\gamma^T + B_n^T (\Phi - I)\right] z_t$$

- Factor loadings in a multifactor Vasicek model:

$$B_{i,n} = 1 + \varphi_i + \varphi_i^2 + \ldots + \varphi_i^n = \sum_{i=1}^{n} \varphi_i^{n-1} = u_1 \times \frac{1 - r^n}{1 - r} = \frac{1 - \varphi_i^n}{1 - \varphi_i}$$

- The one-period forward rate may thus be written as:

$$f_{n,t} = \xi + B_n^T (I - \Phi)\theta - \frac{1}{2}\sum_{i=1}^{k} \left(\lambda_i \sigma_i + \frac{1 - \varphi_i^n}{1 - \varphi_i} \sigma_i \right)^2 + \sum_{i=1}^{k} \left[\varphi_i^n z_{it}\right]$$

- If the factors that determine the dynamics of the yield curve are non-observable and the parameters are unknown, a usual estimation methodology is the Kalman filter and a maximum likelihood procedure.

- <u>Kalman Filter</u> - algorithm that computes the optimal estimate for the state variables at *t* using the information available up to *t*-1.

- The starting point for the derivation of the Kalman filter is to write the model in state-space form:

\- observation or measurement equation

$$\underset{(r\times1)}{Y_t} = \underset{(r\times n)}{A}\cdot\underset{(n\times1)}{X_t} + \underset{(r\times k)}{H}\cdot\underset{(k\times1)}{Z_t} + \underset{(r\times1)}{w_t}$$

\- state or transition equation.

$$\underset{(k\times1)}{Z_t} = \underset{(k\times1)}{C} + \underset{(k\times k)}{F}\cdot\underset{(k\times1)}{Z_{t-1}} + G\,\underset{(k\times1)}{v_t}$$

r – No. variables to estimate
n – No. observable exogenous variables
k – No. non-observable or latent exogenous variables (the factors).

The variance matrices are written as:

$$\underset{(r\times r)}{R} = E(w_t w_t')$$

$$\underset{(k\times k)}{Q} = E(v_{t+1} v_{t+1}')$$

- <u>2-factor model:</u>

$$\begin{bmatrix} y_{1,t} \\ \vdots \\ y_{l,t} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ \vdots \\ a_{l,t} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{2,1} \\ \vdots & \vdots \\ b_{1,l} & b_{2,l} \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} w_{1,t} \\ \vdots \\ w_{l,t} \end{bmatrix}$$

$$\begin{bmatrix} z_{1,t+1} \\ z_{2,t+1} \end{bmatrix} = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix} \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_{1,t+1} \\ v_{2,t+1} \end{bmatrix}$$

2.3 – HJM

Goal: Model the dynamics of the entire yield curve.
The yield curve itself (rather than the short rate *r) is the explanatory variable.*

Proposed by Heath-Jarrow and Morton (1992).

Use the observed yield curve as initial data. ➡ $f^{\star}(0,T)$

Model (instantaneous) forward rates. One SDE for each maturity date *T.* (**infinite dimensional system**)

$$
Q\text{-dynamics:} \qquad \begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t), \\ f(0,T) &= f^{\star}(0,T). \end{aligned}
$$

Main Theorem: (HJM:s drift condition)

Under the martingale measure Q, the following must hold

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds.$$

$$df(t,T) = \boxed{\sigma(t,T)\left(\int_t^T \sigma(t,s)ds\right)} dt + \sigma(t,T)dW(t)$$

- **HJM shows that there is a link between the drift and the standard deviation of the instantaneous forward rate.**

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

$$f(t,T) = \frac{\partial \log p(t,T)}{\partial T}$$

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$$

Specifying forward rates.

Thus: $\Longleftrightarrow$

Specifying bond prices.

- **Problem:** short rate is non-Markovian, i.e. its values depend on the previous poath followed => Monte Carlo simulation or nonrecombining trees have to be used (a binomial tree with 30 steps means 2^{30} nodes, roughly 2B).