PART IV
RISK NEUTRAL DENSITY FUNCTIONS

Risk-Neutral Density Functions

- Prices of financial derivatives reflect the expectations of economic agents regarding the future path of the prices of the underlying assets.
- While forward and futures contracts provide information on the expected value of the prices of the underlying assets, option prices allow the estimation of the RND function of the prices of the underlying assets, giving a more complete picture about the expectations on their future evolution.
- One can derive a relationship between European option prices and the RND, starting by considering the basic pricing equation applied to call-option prices:

$$C(X)_t = E_{\Pi,t}\left[\max(S_T - X, 0) \cdot M_T\right]$$

where *Π* denotes that the expected value is computed using the true or original probability measure *Π*, represented by a density function π.

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- From the consumption-based CAPM, the stochastic discount factor is the nominal intertemporal marginal rate of substitution, denoted by $MRS_{t,T}$, where Q_t is the nominal price index.

$$C(X)_t = E_t\left[\max(S_T - X,0)\left[\delta \frac{U'(C_T)}{U'(C_t)} \Big/ \frac{Q_T}{Q_t}\right]\right]$$
$$= E_t\left[\max(S_T - X,0)\cdot MRS_{t,T}\right]$$

- In order to compute the expected value, one uses the density π_t related to the probability measure Π:

$$C(X)_t = \int_0^\infty \max(S_T - X,0)\cdot MRS_{t,T}\pi_t(S_T)dS_T$$
$$= \int_0^\infty \max(S_T - X,0)\cdot \frac{MRS_{t,T}\pi_t(S_T)}{\int_0^\infty MRS_{t,T}\pi_t(S_T)} e^{-i_t^f\tau} dS_T$$
$$= e^{-i_t^f\tau}\int_0^\infty \max(S_T - X,0)\omega_t(S_T)dS_T$$
$$= e^{-i_t^f\tau}E_{\Omega,t}\max(S_T - X,0)$$

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where i_t^f is the risk-free rate in t for term to maturity τ ($\tau = T - t$) and $\omega_t = \dfrac{MRS_{t,T}\pi_t(S_T)}{\int_0^\infty MRS_{t,T}\pi_t(S_T)}$ is alternatively known as the risk-neutral probability density (RND) associated to the probability measure Ω (see Cox and Ross (1976)), the equivalent martingale measure (see Harrison and Kreps (1979)) or the state-price density (SPD), being, as referred in Ait-Sahalia and Lo (1999), the continuous-state counterpart to the prices of Arrow-Debreu state-contingent claims.[28]

[28] These assets were introduced in economics by Arrow (1964) and Debreu (1959). They are characterised by paying one monetary unit in a given state and nothing in all other states. Probability density functions could be directly obtained from the prices of Arrow-Debreu securities if these were traded for every state.

Risk-Neutral Density Functions

- It can be easily concluded that $\omega_t(S_T)$ is a probability density function, as it assumes values only in the interval between 0 and 1 and its integral is equal to 1.
- Differentiating in order to the strike price, we obtain:

$$\frac{\partial C}{\partial X} = -e^{-i_t^f \tau} \int_X^\infty \omega(S_T) dS_T = $$
$$= -e^{-i_t^f \tau} \left(1 - \int_{-\infty}^X \omega(S_T) dS_T\right)$$

$$\Rightarrow \quad P_\Omega\left[S_T \leq X\right] = 1 + \frac{\partial C}{\partial X} e^{r_\tau \tau}$$

- This function is monotonously increasing and is bounded between 0 and 1, as the call-option price curve is also monotonous and negatively sloped (between –1 and 0).

Risk-Neutral Density Functions

- It assumes higher absolute values at the left tail, when the call-options are deep-in-the-money and their prices are higher, and lower absolute values at the right tail, when the call-options are deep-out-of-the-money and their prices are zero.
- Obviously, the density function will be obtained by the differentiation of the LHS:

$$\omega(X) = e^{i_f \tau} \cdot \frac{\partial^2 C(X)}{\partial X^2}$$

- Identical relationships can be obtained for put-options. As the pay-off of a put-option is $\max(X - S_T, 0)$, the cumulative probability distribution function is given as:

$$P_Q[S_T \leq X] = \frac{\partial P}{\partial X} e^{i_f \tau}$$

Risk-Neutral Density Functions

- Main techniques to estimate the RND functions from European option prices:

(i) non-parametric methods - avoids any kind of parametric specification on the stochastic process of the underlying financial asset price, the option premium function, the implied volatility or even the RND.

(ii) Direct parametric methods - based on assumptions about the stochastic process or the terminal distribution of the underlying asset price => RND parameters are given by minimising the squared difference between observed and estimated option prices.

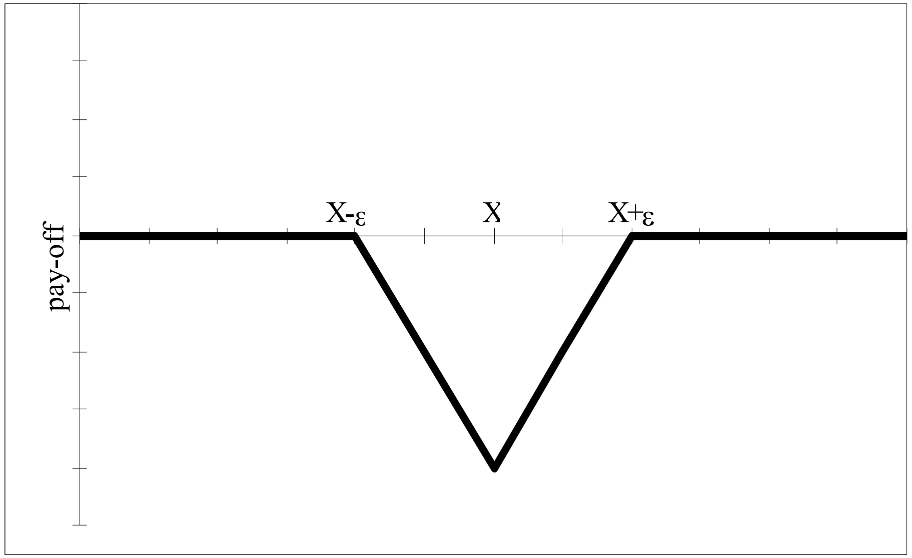
(iii) Indirect parametric methods - assumes a parametric specification for a function that is related to the RND, namely the option price or the implied volatility function.

Risk-Neutral Density Functions

- Non-parametric methods can be implemented directly from the theoretical relationship between option prices and the RND functions, corresponding the latter to the prices of state-contingent claims or Arrow-Debreu securities.
- Though these securities are not usually available in financial markets, one can construct them from the option prices.
- Following Breeden and Litzenberger (1978), a portfolio resulting from buying two call-options with strike price X and selling two call-options, with strike prices $X-\varepsilon$ and $X+\varepsilon$, has a pay-off function usually called butterfly spread.

Risk-Neutral Density Functions

Pay-off function of a butterfly spread



- Price of the symmetric of the butterfly spread:

$$D(X;\varepsilon)=\frac{[C(X+\varepsilon)-C(X)]-[C(X)-C(X-\varepsilon)]}{\varepsilon}$$

$$\lim_{\varepsilon\to 0}\frac{D(X;\varepsilon)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{[C(X+\varepsilon)-C(X)]-[C(X)-C(X-\varepsilon)]}{\varepsilon^2}=\frac{\partial^2 C(X)}{\partial X^2}$$

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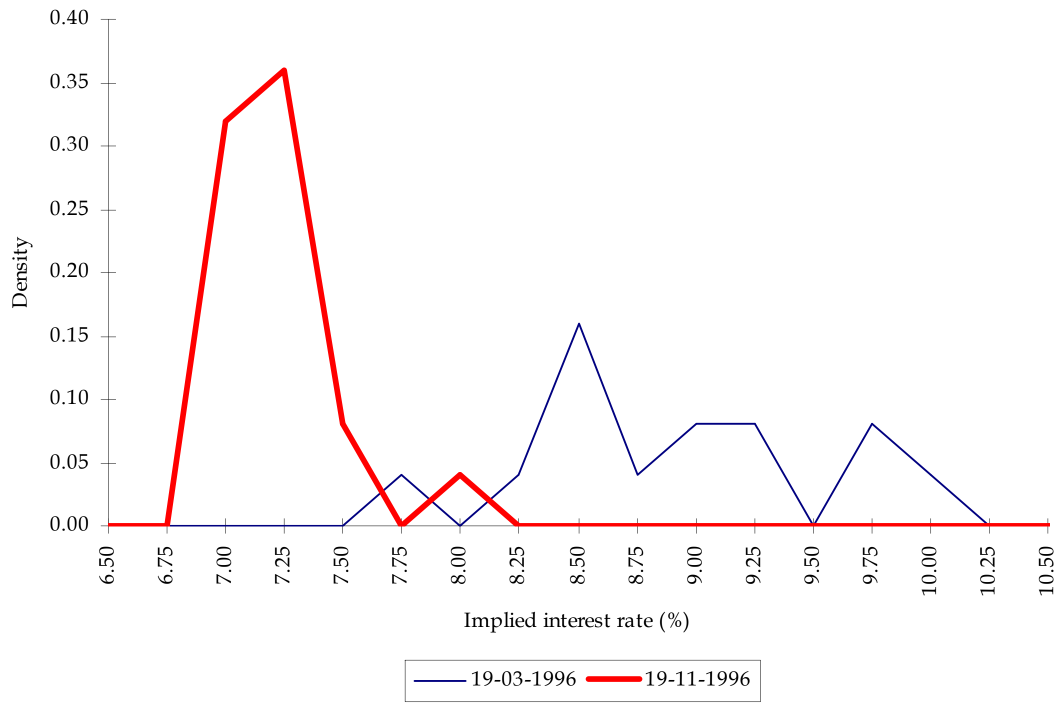
- In derivative exchange traded options, strike prices are spaced by small intervals, though not necessarily close to zero. Thus, a discrete approximation may be used:

$$\frac{\partial^2 C(X)}{\partial X^2} \approx \left[\frac{C(X_{i+1}) - C(X_i)}{X_{i+1} - X_i} - \frac{C(X_i) - C(X_{i-1})}{X_i - X_{i-1}}\right] \Big/ \left[\frac{1}{2}\left(X_{i+1} - X_{i-1}\right)\right]$$

Risk-Neutral Density Functions

- However, density functions obtained with these approaches are frequently too irregular.

RND functions estimated with a discrete approximation: 3-month Italian Lira for 18 December 1996



Risk-Neutral Density Functions

Estimation methodologies – non-parametric methodologies:

- Kernel smoothing - estimate the option prices based on a weighted average of the option prices observed, with the weights decreasing with the distance to the strike price evaluated:

$$\hat{C}(x) = \frac{1}{N}\sum_{i=1}^{N} \omega_{i,N}(x) C^0(X_i)$$

where $\omega_{i,N}(x)$ are the weights attached to the prices $C^0(X_i)$ (with higher weights attached to the $C^0(X_i)$ corresponding to the strikes X_i closer to x). When $\omega_{i,N}(x)$ is a probability density function, this technique is called kernel smoothing.[17]

Risk-Neutral Density Functions

Estimation methodologies –parametric methodologies:

(i) Linear combination of two log-normal distributions

- It consists in solving the following optimisation problem:

$$\underset{\alpha_1,\alpha_2,\beta_1,\beta_2,\theta}{Min} \sum_{i=1}^{N}\left[\hat{C}(X_i,\tau)-C_i^0\right]^2+\sum_{i=1}^{N}\left[\hat{P}(X_i,\tau)-P_i^0\right]^2$$

s.t. $\beta_1,\beta_2>0$ and $0\leq\theta\leq 1$.

$$\hat{C}(X_i,\tau)=e^{-r\tau}\int_{X_i}\left[\theta L(\alpha_1,\beta_1;S_T)+(1-\theta)L(\alpha_2,\beta_2;S_T)\right](S_T-X_i)dS_T$$

$$=e^{-r\tau}\theta\left[e^{\alpha_1+\frac{1}{2}\beta_1^2}N\left(\frac{-\ln(X_i)+(\alpha_1+\beta_1^2)}{\beta_1}\right)-X_iN\left(\frac{-\ln(X_i)+\alpha_1}{\beta_1}\right)\right]+$$

$$=e^{-r\tau}(1-\theta)\left[e^{\alpha_2+\frac{1}{2}\beta_2^2}N\left(\frac{-\ln(X_i)+(\alpha_2+\beta_2^2)}{\beta_2}\right)-X_iN\left(\frac{-\ln(X_i)+\alpha_2}{\beta_2}\right)\right]$$

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Estimation methodologies –parametric methodologies:

$$\hat{P}(X_i,\tau) = e^{-r\tau}\int_{-\infty}^{X_i}\left[\theta L(\alpha_1,\beta_1;S_T)+(1-\theta)L(\alpha_2,\beta_2;S_T)\right](X_i - S_T)dS_T$$

$$= e^{-r\tau}\theta\left[-e^{\alpha_1+\frac{1}{2}\beta_1^2}N\left(\frac{\ln(X_i)-(\alpha_1+\beta_1^2)}{\beta_1}\right)+X_i N\left(\frac{\ln(X_i)-\alpha_1}{\beta_1}\right)\right]+$$

$$= e^{-r\tau}(1-\theta)\left[-e^{\alpha_2+\frac{1}{2}\beta_2^2}N\left(\frac{\ln(X_i)-(\alpha_2+\beta_2^2)}{\beta_2}\right)+X_i N\left(\frac{\ln(X_i)-\alpha_2}{\beta_2}\right)\right]$$

and where $L(\alpha_i,\beta_i;S_T)$ is the log-normal density function i (i = 1, 2), the parameters α_1 and α_2 are the means of the respective normal distributions, β_1 and β_2 are the standard-deviations of the latter and θ the weight attached to each distribution. The expressions for α_i and β_i are the following:

$$\alpha_i = \ln F_t + \left(\mu_i - \frac{\sigma_i^2}{2}\right)\tau$$

$$\beta_i = \sigma_i\sqrt{\tau}$$

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Estimation methodologies –parametric methodologies:

- This technique is due to Ritchey (1990) and Melick and Thomas (1997).

- Though this method imposes some structure on the density function and raises some empirical difficulties, it offers some advantages, as it is sufficiently flexible and fast.

Risk-Neutral Density Functions

Estimation methodologies –parametric methodologies:

(ii) Fitting of the volatility smile

- One can also estimate the volatility smile (relationship between implied vols and strike prices, namely through a polynomial adjustment.
- After estimating the vols, one can calculate the corresponding option prices.
- From these, the RND can be computed directly.