Arbitrage Theory in Continuous Time

Exercises

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INTRODUCTION

No exercises for this chapter.

2

THE BINOMIAL MODEL

2.1 Exercises

Exercise 2.1

(a) Prove Proposition 2.6.

(b) Show, in the one period binomial model, that if $\Pi(1; X) \neq X$ with probability 1, then you can make a riskless profit.

Exercise 2.2 Prove Proposition 2.21.

Exercise 2.3 Consider the multiperiod example in the text. Suppose that at time $t = 1$ the stock price has gone up to 120, and that the market price of the option turns out to be 50.0. Show explictly how you can make an arbitrage profit.

Exercise 2.4 Prove Proposition 2.24, by using induction on the time horizon T.

3

A MORE GENERAL ONE PERIOD MODEL

3.1 Exercises

Exercise 3.1 Prove that Q in Proposition 3.11 is a martingale measure also for the price process $\Pi(t; X)$, i.e. show that

$$\frac{\Pi(0; X)}{B_0} = E^Q\left[\frac{\Pi(1; X)}{B_1}\right].$$

where B is the risk free asset.

Exercise 3.2 Prove the last item in Proposition 3.15.

Exercise 3.3 Prove Proposition 3.18.

4

STOCHASTIC INTEGRALS

4.1 Exercises

Exercise 4.1 Compute the stochastic differential dZ when

(a) $Z(t) = e^{\alpha t}$,

(b) $Z(t) = \int_0^t g(s)dW(s)$, where g is an adapted stochastic process.

(c) $Z(t) = e^{\alpha W(t)}$

(d) $Z(t) = e^{\alpha X(t)}$, where X has the stochastic differential

$$dX(t) = \mu dt + \sigma dW(t)$$

(μ and σ are constants).

(e) $Z(t) = X^2(t)$, where X has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).$$

Exercise 4.2 Compute the stochastic differential for Z when $Z(t) = \frac{1}{X(t)}$ and X has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t).$$

By using the definition $Z = X^{-1}$ you can in fact express the right hand side of dZ entirely in terms of Z itself (rather than in terms of X). Thus Z satisfies a stochastic differential equation. Which one?

Exercise 4.3 Let $\sigma(t)$ be a given deterministic function of time and define the process X by

$$X(t) = \int_0^t \sigma(s)dW(s). \tag{4.1}$$

Use the technique described in Example **??** in order to show that the characteristic function of $X(t)$ (for a fixed t) is given by

$$E\left[e^{iuX(t)}\right] = \exp\left\{-\frac{u^2}{2}\int_0^t \sigma^2(s)ds\right\}, \quad u \in R, \tag{4.2}$$

thus showing that $X(t)$ is normally distibuted with zero mean and a variance given by

$$Var[X(t)] = \int_0^t \sigma^2(s)ds.$$

Exercise 4.4 Suppose that X has the stochastic differential

$$dX(t) = \alpha X(t)dt + \sigma(t)dW(t),$$

where α is a real number whereas $\sigma(t)$ is any stochastic process. Use the technique in Example **??** in order to determine the function $m(t) = E[X(t)]$.

Exercise 4.5 Suppose that the process X has a stochastic differential

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

and that $\mu(t) \geq 0$ with probability one for all t. Show that this implies that X is a submartingale.

Exercise 4.6 A function $h(x_1, \ldots, x_n)$ is said to be **harmonic** if it satisfies the condition

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} = 0.$$

It is **subharmonic** if it satisfies the condition

$$\sum_{i=1}^{n} \frac{\partial^2 h}{\partial x_i^2} \geq 0.$$

Let $W_1, \ldots, W_n$ be independent standard Wiener processes, and define the process X by $X(t) = h(W_1(t), \ldots, W_n(t))$. Show that X is a martingale (submartingale) if h is harmonic (subharmonic).

Exercise 4.7 The object of this exercise is to give an argument for the formal identity

$$dW_1 \cdot dW_2 = 0,$$

when W_1 and W_2 are independent Wiener processes. Let us therefore fix a time t, and divide the interval $[0, t]$ into equidistant points $0 = t_0 < t_1 < \cdots < t_n = t$, where $t_i = \frac{i}{n} \cdot t$. We use the notation

$$\Delta W_i(t_k) = W_i(t_k) - W_i(t_{k-1}), \quad i = 1, 2.$$

Now define Q_n by

$$Q_n = \sum_{k=1}^{n} \Delta W_1(t_k) \cdot \Delta W_2(t_k).$$

Show that $Q_n \to 0$ in L^2, i.e. show that

$$\begin{aligned} E[Q_n] &= 0, \\ Var[Q_n] &\to 0. \end{aligned}$$

Exercise 4.8 Let X and Y be given as the solutions to the following system of stochastic differential equations.

$$\begin{aligned} dX &= \alpha X dt - Y dW, \quad X(0) = x_0, \\ dY &= \alpha Y dt + X dW, \quad Y(0) = y_0. \end{aligned}$$

Note that the initial values $x_0,\ y_0$ are deterministic constants.

(a) Prove that the process R defined by $R(t) = X^2(t) + Y^2(t)$ is deterministic.

(b) Compute $E[X(t)]$.

Exercise 4.9 For a $n \times n$ matrix A, the **trace** of A is defined as

$$tr(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) If B is $n \times d$ and C is $d \times n$, then BC is $n \times n$. Show that

$$tr(BC) = \sum_{ij} B_{ij} C_{ji}.$$

(b) With assumptions as above, show that

$$tr(BC) = tr(CB).$$

(c) Show that the Itô formula in Theorem **??** can be written

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} tr\left[\sigma^\star H \sigma\right] \right\} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma_i dW_i$$

where H denotes the Hessian matrix

$$H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Exercise 4.10 Prove all claims in Section 2.8.

5

DIFFERENTIAL EQUATIONS

5.1 Exercises

Exercise 5.1 Show that the scalar SDE

$$\begin{aligned} dX_t &= \alpha X_t dt + \sigma dW_t, \\ X_0 &= x_0, \end{aligned}$$

has the solution

$$X(t) = e^{\alpha t} \cdot x_0 + \sigma \int_0^t e^{\alpha(t-s)} dW_s, \tag{5.1}$$

by differentiating X as defined by eqn (5.1) and showing that X so defined actually satisfies the SDE.

Hint: Write eqn (5.1) as

$$X_t = Y_t + Z_t \cdot R_t,$$

where

$$\begin{aligned} Y_t &= e^{\alpha t} \cdot x_0, \\ Z_t &= e^{\alpha t} \cdot \sigma, \\ R_t &= \int_0^t e^{-\alpha s} dW_s, \end{aligned}$$

and first compute the differentials dZ, dY and dR. Then use the multidimensional Itô formula on the function $f(y, z, r) = y + z \cdot r$.

Exercise 5.2 Let A be an $n \times n$ matrix, and define the matrix exponential e^A by the series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

This series can be shown to converge uniformly.

(a) Show, by taking derivatives under the summation sign, that

$$\frac{de^{At}}{dt} = Ae^{At}.$$

(b) Show that

$$e^0 = I,$$

where 0 denotes the zero matrix, and I denotes the identity matrix.

(c) Convince yourself that if A and B commute, i.e. $AB = BA$, then

$$e^{A+B} = e^A \cdot e^B = e^B \cdot e^A.$$

Hint: Write the series expansion in detail.

(d) Show that e^A is invertible for every A, and that in fact

$$\left[e^A\right]^{-1} = e^{-A}.$$

(e) Show that for any A, t and s

$$e^{A(t+s)} = e^{At} \cdot e^{As}$$

(f) Show that

$$\left(e^A\right)^\star = e^{A^\star}$$

Exercise 5.3 Use the exercise above to complete the details of the proof of Proposition 5.3.

Exercise 5.4 Consider again the linear SDE (**??**). Show that the expected value function $m(t) = E[X(t)]$, and the covariance matrix $C(t) = \{Cov(X_i(t), X_j(t)\}_{i,j}$ are given by

$$m(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds,$$
$$C(t) = \int_0^t e^{A(t-s)}\sigma(s)\sigma^\star(s)e^{A^\star(t-s)}ds,$$

where * denotes transpose.

Hint: Use the explicit solution above, and the fact that

$$C(t) = E\left[X_t X_t^\star\right] - m(t)m^\star(t).$$

Geometric Brownian motion (GBM) constitutes a class of processes which is closed under a number of nice operations. Here are some examples.

Exercise 5.5 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t.$$

Now define Y by $Y_t = X_t^\beta$, where β is a real number. Then Y is also a GBM process. Compute dY and find out which SDE Y satisfies.

Exercise 5.6 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where V is a Wiener process which is independent of W. Define Z by $Z = \frac{X}{Y}$ and derive an SDE for Z by computing dZ and substituting Z for $\frac{X}{Y}$ in the right hand side of dZ. If X is nominal income and Y describes inflation then Z describes real income.

Exercise 5.7 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dW_t.$$

Note that now both X and Y are driven by the same Wiener process W. Define Z by $Z = \frac{X}{Y}$ and derive an SDE for Z.

Exercise 5.8 Suppose that X satisfies the SDE

$$dX_t = \alpha X_t dt + \sigma X_t dW_t,$$

and Y satisfies

$$dY_t = \gamma Y_t dt + \delta Y_t dV_t,$$

where V is a Wiener process which is independent of W. Define Z by $Z = X \cdot Y$ and derive an SDE for Z. If X describes the price process of, for example, IBM in US$ and Y is the currency rate SEK/US$ then Z describes the dynamics of the IBM stock expressed in SEK.

Exercise 5.9 Use a stochastic representation result in order to solve the following boundary value problem in the domain $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} &= 0, \\ F(T, x) &= \ln(x^2). \end{aligned}$$

Here μ and σ are assumed to be known constants.

Exercise 5.10 Consider the following boundary value problem in the domain $[0, T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t, x) \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t, x) \frac{\partial^2 F}{\partial x^2} + k(t, x) &= 0, \\ F(T, x) &= \Phi(x). \end{aligned}$$

Here μ, σ, k and Φ are assumed to be known functions.

Prove that this problem has the stochastic representation formula

$$F(t,x) = E_{t,x}\left[\Phi(X_T)\right] + \int_t^T E_{t,x}\left[k(s,X_s)\right] ds,$$

where as usual X has the dynamics

$$\begin{aligned} dX_s &= \mu(s,X_s)ds + \sigma(s,X_s)dW_s, \\ X_t &= x. \end{aligned}$$

Hint: Define X as above, assume that F actually solves the PDE and consider the process $Z_s = F(s, X_s)$.

Exercise 5.11 Use the result of the previous exercise in order to solve

$$\begin{aligned} \frac{\partial F}{\partial t} + \frac{1}{2}x^2\frac{\partial^2 F}{\partial x^2} + x &= 0, \\ F(T,x) &= \ln(x^2). \end{aligned}$$

Exercise 5.12 Consider the following boundary value problem in the domain $[0,T] \times R$.

$$\begin{aligned} \frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + r(t,x)F &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

Here $\mu(t,x)$, $\sigma(t,x)$, $r(t,x)$ and $\Phi(x)$ are assumed to be known functions. Prove that this problem has a stochastic representation formula of the form

$$F(t,x) = E_{t,x}\left[\Phi(X_T)e^{\int_t^T r(s,X_s)ds}\right],$$

by considering the process $Z_s = F(s,X_s) \times \exp\left[\int_t^s r(u,X_u)du\right]$ on the time interval $[t,T]$.

Exercise 5.13 Solve the boundary value problem

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x,y) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\delta^2\frac{\partial^2 F}{\partial y^2}(t,x,y) &= 0, \\ F(T,x,y) &= xy. \end{aligned}$$

Exercise 5.14 Go through the details in the derivation of the Kolmogorov forward equation.

Exercise 5.15 Consider the SDE

$$dX_t = \alpha dt + \sigma dW_t,$$

where α and σ are constants.

(a) Compute the transition density $p(s,y;t,x)$, by solving the SDE.

(b) Write down the Fokker–Planck equation for the transition density and check the equation is indeed satisfied by your answer in (a).

Exercise 5.16 Consider the standard GBM

$$dX_t = \alpha X_t dt + \sigma X_t dW_t$$

and use the representation

$$X_t = X_s \exp\left\{\left[\alpha - \frac{1}{2}\sigma^2\right](t-s) + \sigma\left[W_t - W_s\right]\right\}$$

in order to derive the transition density $p(s, y; t, x)$ of GBM. Check that this density satisfies the Fokker-Planck equation in Example 5.14.

6

PORTFOLIO DYNAMICS

6.1 Exercises

Exercise 6.1 Work out the details in the derivation of the dynamics of a self-financing portfolio in the dividend paying case.

7

ARBITRAGE PRICING

7.1 Exercises

Exercise 7.1 Consider the standard Black–Scholes model and a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(S(T))$. Denote the corresponding arbitrage free price process by $\Pi(t)$.

(a) Show that, under the martingale measure Q, $\Pi(t)$ has a local rate of return equal to the short rate of interest r. In other words show that $\Pi(t)$ has a differential of the form

$$d\Pi(t) = r \cdot \Pi(t)\, dt + g(t)dW(t).$$

Hint: Use the Q-dynamics of S together with the fact that F satisfies the pricing PDE.

(b) Show that, under the martingale measure Q, the process $Z(t) = \frac{\Pi(t)}{B(t)}$ is a **martingale**. More precisely, show that the stochastic differential for Z has zero drift term, i.e. it is of the form

$$dZ(t) = Z(t)\sigma_Z(t)dW(t).$$

Determine also the diffusion process $\sigma_Z(t)$ (in terms of the pricing function F and its derivatives).

Exercise 7.2 Consider the standard Black–Scholes model. An innovative company, *F&H INC*, has produced the derivative "the Golden Logarithm", henceforth abbreviated as the GL. The holder of a GL with maturity time T, denoted as $GL(T)$, will, at time T, obtain the sum $\ln S(T)$. Note that if $S(T) < 1$ this means that the holder has to pay a positive amount to *F&H INC*. Determine the arbitrage free price process for the $GL(T)$.

Exercise 7.3 Consider the standard Black–Scholes model. Derive the Black–Scholes formula for the European call option.

Exercise 7.4 Consider the standard Black–Scholes model. Derive the arbitrage free price process for the T-claim $\mathcal{X}$ where $\mathcal{X}$ is given by $\mathcal{X} = \{S(T)\}^{\beta}$. Here β is a known constant.

Hint: For this problem you may find Exercises 5.5 and 4.4 useful.

Exercise 7.5 A so called **binary option** is a claim which pays a certain amount if the stock price at a certain date falls within some prespecified interval. Otherwise nothing will be paid out. Consider a binary option which pays K SEK to the holder at date T if the stock price at time T is in the inerval $[\alpha, \beta]$. Determine the arbitrage free price. The pricing formula will involve the standard Gaussian cumulative distribution function N.

Exercise 7.6 Consider the standard Black–Scholes model. Derive the arbitrage free price process for the claim $\mathcal{X}$ where $\mathcal{X}$ is given by $\mathcal{X} = \frac{S(T_1)}{S(T_0)}$. The times T_0 and T_1 are given and the claim is paid out at time T_1.

Exercise 7.7 Consider the American corporation *ACME INC*. The price process S for *ACME* is of course denoted in US$ and has the P-dynamics

$$dS = \alpha S dt + \sigma S d\bar{W}_1,$$

where α and σ are known constants. The currency ratio SEK/US$ is denoted by Y and Y has the dynamics

$$dY = \beta Y dt + \delta Y d\bar{W}_2,$$

where $\bar{W}_2$ is independent of $\bar{W}_1$. The broker firm *F&H* has invented the derivative "Euler". The holder of a T-Euler will, at the time of maturity T, obtain the sum

$$\mathcal{X} = \ln\left[\{Z(T)\}^2\right]$$

in SEK. Here $Z(t)$ is the price at time t in SEK of the *ACME* stock.

Compute the arbitrage free price (in SEK) at time t of a T-Euler, given that the price (in SEK) of the *ACME* stock is z. The Swedish short rate is denoted by r.

Exercise 7.8 Prove formula (7.52).

Exercise 7.9 Derive a formula for the value, at s, of a forward contract on the T-claim X, where the forward contract is made at t, and $t < s < T$.

8

COMPLETENESS AND HEDGING

8.1 Exercises

Exercise 8.1 Consider a model for the stock market where the short rate of interest r is a deterministic constant. We focus on a particular stock with price process S. Under the objective probability measure P we have the following dynamics for the price process.

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + \delta S(t-)dN(t).$$

Here W is a standard Wiener process whereas N is a Poisson process with intensity λ. We assume that α, σ, δ and λ are known to us. The dN term is to be interpreted in the following way.

- Between the jump times of the Poisson process N, the S-process behaves just like ordinary geometric Brownian motion.

- If N has a jump at time t this induces S to have a jump at time t. The size of the S-jump is given by

$$S(t) - S(t-) = \delta \cdot S(t-).$$

Discuss the following questions.

(a) Is the model free of arbitrage?

(b) Is the model complete?

(c) Is there a unique arbitrage free price for, say, a European call option?

(d) Suppose that you want to replicate a European call option maturing in January 1999. Is it posssible (theoretically) to replicate this asset by a portfolio consisting of bonds, the underlying stock and European call option maturing in December 2001?

Exercise 8.2 Use the Feynman–Kač technique in order to derive a risk neutral valuation formula in connection with Proposition 8.6.

Exercise 8.3 The fairly unknown company *F&H INC.* has blessed the market with a new derivative, "the Mean". With "effective period" given by $[T_1, T_2]$ the holder of a Mean contract will, at the date of maturity T_2, obtain the amount

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)du.$$

Determine the arbitrage free price, at time t, of the Mean contract. Assume that you live in a standard Black–Scholes world, and that $t < T_1$.

Exercise 8.4 Consider the standard Black–Scholes model, and n different simple contingent claims with contract functions $\Phi_1, \ldots, \Phi_n$. Let

$$V = \sum_{i=1}^{n} h_i(t) S_i(t)$$

denote the value process of a self-financing, Markovian portfolio, i.e a portfolio of the form $h(t, S_t)$. Because of the Markovian assumption, V will be of the form $V(t, S_t)$. Show that V satisfies the Black–Scholes equation.

9

PARITY RELATIONS AND DELTA HEDGING

9.1 Exercises

Exercise 9.1 Consider the standard Black–Scholes model. Fix the time of maturity T and consider the following T-claim $\mathcal{X}$.

$$\mathcal{X} = \begin{cases} K & \text{if } S(T) \leq A \\ K + A - S(T) & \text{if } A < S(T) < K + A \\ 0 & \text{if } S(T) > K + A. \end{cases} \tag{9.1}$$

This contract can be replicated using a portfolio, consisting solely of bonds, stock and European call options, which is constant over time. Determine this portfolio as well as the arbitrage free price of the contract.

Exercise 9.2 The setup is the same as in the previous exercise. Here the contract is a so called

$$\mathcal{X} = \begin{cases} K - S(T) & \text{if } 0 < S(T) \leq K \\ S(T) - K & \text{if } K < S(T). \end{cases} \tag{9.2}$$

Determine the constant replicating portfolio as well as the arbitrage free price of the contract.

Exercise 9.3 The setup is the same as in the previous exercises. We will now study a so called "bull spread" (see Fig. 9.1). With this contract we can, to a limited extent, take advantage of an increase in the market price while being protected from a decrease. The contract is defined by

$$\mathcal{X} = \begin{cases} B & \text{if } S(T) > B \\ S(T) & \text{if } A \leq S(T) \leq B \\ A & \text{if } S(T) < A. \end{cases} \tag{9.3}$$

FIG. 9.1.

We have of course the relation $A < B$. Determine the constant replicating portfolio as well as the arbitrage free price of the contract.

Exercise 9.4 The setup and the problem are the same as in the previous exercises. The contract is defined by

$$\mathcal{X} = \begin{cases} 0 & \text{if } S(T) < A \\ S(T) - A & \text{if } A \leq S(T) \leq B \\ C - S(T) & \text{if } B \leq S(T) \leq C \\ 0 & \text{if } S(T) > C. \end{cases} \tag{9.4}$$

By definition the point C divides the interval $[A, C]$ in the middle, i.e $B = \frac{A+C}{2}$.

Exercise 9.5 Suppose that you have a portfolio P with $\Delta_P = 2$ and $\Gamma_P = 3$. You want to make this portfolio delta and gamma neutral by using two derivatives F and G, with $\Delta_F = -1$, $\Gamma_F = 2$, $\Delta_G = 5$ and $\Gamma_G = -2$. Compute the hedge.

Exercise 9.6 Consider the same situation as above, with the difference that now you want to use the underlying S instead of G. Construct the hedge according to the two step scheme descibed in Section 9.6.

Exercise 9.7 Prove Proposition 9.7 by comparing the stock holdings in the continuously rebalanced portfolio to the replicating portfolio in Theorem 8.5 of the previous chapter.

Exercise 9.8 Consider a self-financing Markovian portfolio (in continuous time) containing various derivatives of the single underlying asset in the Black–Scholes model. Denote the value (pricing function) of the portfolio by $P(t, s)$. Show that the following relation must hold between the various greeks of P.

$$\Theta_P + rs\Delta_P + \frac{1}{2}\sigma^2 s^2 \Gamma_P = rP.$$

Hint: Use Exercise 8.4.

Exercise 9.9 Use the result in the previous exercise to show that if the portfolio is both delta and gamma neutral, then it replicates the risk free asset, i.e. it has a risk free rate of return which is equal to the short rate r.

Exercise 9.10 Show that for a European put option the delta and gamma are given by

$$\begin{aligned} \Delta &= N[d_1] - 1, \\ \Gamma &= \frac{\varphi(d_1)}{s\sigma\sqrt{T-t}}. \end{aligned}$$

Hint: Use put–call parity.

Exercise 9.11 Take as given the usual portfolio P, and investigate how you can hedge it in order to make it both delta and vega neutral.

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THE MARTINGALE APPROACH TO ARBITRAGE THEORY*

No exercises for this chapter.

11

THE MATHEMATICS OF THE MARTINGALE APPROACH*

11.1 Exercises

Exercise 11.1 Complete an argument in the proof of Theorem 11.1 by proving that if X and Y are random variables of the form

$$X = x_0 + \int_0^T g_s dW_s,$$
$$Y = y_0 + \int_0^T h_s dW_s,$$

and if g and h have disjoint support on the time axis, i.e. if

$$g_t h_t = 0, \quad P - a.s. \quad 0 \leq t \leq T$$

then

$$XY = x_0 y_0 + \int_0^T [X_s h_s + Y_s g_s] \, dW_s.$$

Hint: Define the processes X_t and Y_t by $X_t = x_0 + \int_0^t g_s dW_s$ and correspondingly for Y and use the Itô formula.

Exercise 11.2 Consider the following SDE.

$$dX_t = \alpha f(X_t)dt + \sigma(X_t)dW_t,$$
$$X_0 = x_0.$$

Here f and σ are known functions, whereas α is an unknown parameter. We assume that the SDE possesses a unique solution for every fixed choice of α.

Construct a dynamical statistical model for this problem and compute the ML estimator process $\hat{\alpha}_t$ for α, based upon observations of X.

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BLACK-SCHOLES FROM A MARTINGALE POINT OF VIEW*

No exercises for this chapter.

13

MULTIDIMENSIONAL MODELS: CLASSICAL APPROACH

13.1 Exercises

Exercise 13.1 Prove Proposition 13.3.

Exercise 13.2 Check all calculations in the derivation of the PDE in Proposition 13.4.

Exercise 13.3 Consider again the exchange option in Example 13.5. Now assume that $\bar{W}_1$ and $\bar{W}_2$ are no longer independent, but that the local correlation is given by $d\bar{W}_1 \cdot d\bar{W}_2 = \rho dt$. (We still assume that both Wiener processes have unit variance parameter, i.e. that $d\bar{W}_1^2 = d\bar{W}_2^2 = dt$.) How will this affect the Black–Scholes-type formula given in the example?

Exercise 13.4 Consider the stock price model in Example 13.5. The T-contract $\mathcal{X}$ to be priced is defined by

$$\mathcal{X} = \max\left[aS_1(T), bS_2(T)\right],$$

where a and b are given positive numbers. Thus, up to the scaling factors a and b, we obtain the maximum of the two stock prices at time T. Use Proposition 13.4 and the Black–Scholes formula in order to derive a pricing formula for this contract. See Johnson (1987).

Hint: You may find the following formula (for $x > 0$) useful.

$$\max[x, 1] = 1 + \max[x - 1, 0].$$

Exercise 13.5 Use the ideas in Section 13.4 to analyze the pricing PDE for a claim of the form $\mathcal{X} = \Phi(S(T))$ where we now assume that Φ is homogeneous of degree β, i.e.

$$\Phi(t \cdot s) = t^{\beta}\Phi(s), \quad \forall t > 0.$$

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MULTIDIMENSIONAL MODELS: MARTINGALE APPROACH*

14.1 Exercises

Exercise 14.1 Derive (14.46).

Exercise 14.2 Assume generic absence of arbitrage and prove that any market price of risk process λ generating a martingale measure must be of the form

$$\lambda(t) = \hat{\lambda}(t) + \mu(t)$$

where $\mu(t)$ is orthogonal to the rows of $\sigma(t)$ for all t.

15

INCOMPLETE MARKETS

15.1 Exercises

Exercise 15.1 Consider a claim $\Phi(X(T))$ with pricing function $F(t,x)$. Prove Proposition 15.4, i.e. prove that dF under Q has the form

$$dF = rF dt + \{\cdots\} dW,$$

where W is a Q-Wiener process.

Hint: Use Itô's formula on F, using the Q-dynamics of X. Then use the fact that F satisfies the pricing PDE.

Exercise 15.2 Convince yourself, either in the scalar or in the multidimensional case, that the market price of risk process λ really is of the form

$$\lambda = \lambda(t, X(t)).$$

Exercise 15.3 Prove Proposition 15.7.

Exercise 15.4 Consider the scalar model in Section 15.2 and a fixed claim $\Gamma(X(T))$. Take as given a pricing function $G(t,x)$, for this claim, satisfying the boundary condition $G(T,x) = \Gamma(x)$, and assume that the corresponding volatility function $\sigma_G(t,x)$ is nonzero. We now expect the market $[B, G]$ to be complete. Show that this is indeed the case, i.e. show that every simple claim of the form $\Phi(X(T))$ can be replicated by a portfolio based on B and G.

Exercise 15.5 Consider the multidimensional model in Section 15.3 and a fixed family of claims $\Phi_i(X(T)),\ i = 1,\dots,n$. Take as given a family of pricing functions $F^i(t,x),\ i = 1,\dots,n$, for these claims, satisfying the boundary condition $F^i(T,x) = \Phi_i(x),\ i = 1,\dots,n$, and assume that the corresponding volatility matrix $\sigma(t,x)$ is nonzero. Show that the market $[B, F^1,\dots,F^n]$ is complete, i.e. show that every simple claim of the form $\Phi(X(T))$ can be replicated by a portfolio based on $[B, F^1,\dots,F^n]$.

Exercise 15.6 Prove the propositions in Section 15.4.

16

DIVIDENDS

16.1 Exercises

Exercise 16.1 Prove Proposition 16.1. Assume that you are standing at $t-$ and that the conclusion of the theorem does not hold. Show that by trading at $t-$ and t you can then create an arbitrage. This is mathematically slightly imprecise, and the advanced reader is invited to provide a precise proof based on the martingale approach of Chapter 10.

Exercise 16.2 Prove the cost of carry formula (16.25).

Exercise 16.3 Derive a cost-of-carry formula for the case of discrete dividends.

Exercise 16.4 Prove Proposition 16.9.

Exercise 16.5 Prove Proposition 16.10.

Exercise 16.6 Consider the Black–Scholes model with a constant continuous dividend yield δ. Prove the following put–call parity relation, where c_δ (p_δ) denotes the price of a European call (put).

$$p_\delta = c_\delta - se^{-\delta(T-t)} + Ke^{-r(T-t)}.$$

Exercise 16.7 Consider the Black–Scholes model with a constant discrete dividend, as in eqns (16.17)–(16.18). Derive the relevant put–call parity for this case, given that there are n remaining dividend points.

Exercise 16.8 Consider the Black–Scholes model with a constant continuous dividend yield δ. The object of this exercise is to show that this model is complete. Take therefore as given a contingent claim $\mathcal{X} = \Phi(S(T))$. Show that this claim can be replicated by a self-financing portfolio based on B and S, and that the portfolios weights are given by

$$u^B(t,s) = \frac{F(t,s) - sF_s(t,s)}{F(t,s)},$$
$$u^S(t,s) = \frac{sF_s(t,s)}{F(t,s)},$$

where F is the solution of the pricing eqn (16.8).

Hint: Copy the reasoning from Chapter 8, while using the self-financing dynamics given in Section 6.3.

Exercise 16.9 Consider the Black–Scholes model with a constant continuous dividend yield δ. Use the result from the previous exercise in order to compute explicitly the replicating portfolio for the claim $\Phi(S(T)) = S(T)$.

Exercise 16.10 Check that, when $\gamma = 0$ in Section 16.2.2, all results degenerate into those of Section 16.2.1.

Exercise 16.11 Prove Propositions 16.11-16.13.

17

CURRENCY DERIVATIVES

17.1 Exercises

Exercise 17.1 Consider the European call on the exchange rate described at the end of Section 17.1. Denote the price of the call by $c(t, x)$, and denote the price of the corresponding put option (with the same exercise price K and exercise date T) by $p(t, x)$. Show that the put–call parity relation between p and c is given by

$$p = c - xe^{-r_f(T-t)} + Ke^{-r_d(T-t)}$$

Exercise 17.2 Compute the pricing function (in the domestic currency) for a **binary option** on the exchange rate. This option is a T-claim, $\mathcal{Z}$, of the form

$$\mathcal{Z} = 1_{[a,b]}(X(T)),$$

i.e. if $a \leq X(T) \leq b$ then you will obtain one unit of the domestic currency, otherwise you get nothing.

Exercise 17.3 Derive the dynamics of the domestic stock price S_d under the foreign martingale measure Q_f.

Exercise 17.4 Compute a pricing formula for the exchange option in (17.19). Use the ideas from Section 13.4 in order to reduce the complexity of the formula. For simplicity you may assume that the processes S_d, S_f and X are uncorrelated.

Exercise 17.5 Consider a model with the domestic short rate r_d and two foreign currencies, the exchange rates of which (from the domestic perspective) are denoted by X_1 and X_2 respectively. The foreign short rates are denoted by r_1 and r_2 respectively. We assume that the exchange rates have P-dynamics given by

$$dX_i = X_i \alpha_i dt + X_i \sigma_i d\bar{W}_i, \quad i = 1, 2,$$

where $\bar{W}_1, \bar{W}_2$ are P-Wiener processes with correlation ρ.

(a) Derive the pricing PDE for contracts, quoted in the domestic currency, of the form $\mathcal{Z} = \Phi(X_1(T), X_2(T))$.

(b) Derive the corresponding risk neutral valuation formula, and the Q_d-dynamics of X_1 and X_2.

(c) Compute the price, in domestic terms, of the "binary quanto contract" $\mathcal{Z}$, which gives, at time T, K units of foreign currency No. 1, if $a \leq X_2(T) \leq b$, (where a and b are given numbers), and zero otherwise. If you want to facilitate computations you may assume that $\rho = 0$.

Exercise 17.6 Consider the model of the previous exercise. Compute the price, in domestic terms, of a quanto exchange option, which at time T gives you the option, but not the obligation, to exchange K units of currency No. 1 for 1 unit of currency No. 2.

Hint: It is possible to reduce the state space as in Section 13.4.

18

BARRIER OPTIONS

18.1 Exercises

In all exercises below we assume a standard Black–Scholes model.

Exercise 18.1 An "all-or-nothing" contract, with delivery date T, and strike price K, will pay you the amount K, if the price of the underlying stock exceeds the level L at some time during the interval $[0, t]$. Otherwise it will pay nothing. Compute the price, at $t < T$, of the all-or-nothing contract. In order to avoid trivialities, we assume that $S(s) < L$ for all $s \leq t$.

Exercise 18.2 Consider a binary contract, i.e. a T-claim of the form

$$\mathcal{X} = I_{[a,b]}(S_T),$$

where as usual I is the indicator function. Compute the price of the down-and-out version of the binary contract above, for all possible values of the barrier L.

Exercise 18.3 Consider a general down-and-out contract, with contract function Φ, as descibed in Section 18.2.1. We now modify the contract by adding a fixed "rebate" A, and the entire contract is specified as follows.

If $S(t) > L$ for all $t \leq T$ then $\Phi(S(T))$ is paid to the holder.

If $S(t) \leq L$ for some $t \leq T$ then the holder receives the fixed amount A.

Derive a pricing formula for this contract.

Hint: Use Proposition 18.4.

Exercise 18.4 Use the exercise above to price a down-and-out European call with rebate A.

Exercise 18.5 Derive a pricing formula for a down-and-out version of the T contract $\mathcal{X} = \Phi(S(T))$, when S has a continuous dividend yield δ. Specialize to the case of a European call.

19

STOCHASTIC OPTIMAL CONTROL

19.1 Exercises

Exercise 19.1 Solve the problem of maximizing logarithmic utility

$$E\left[\int_0^T e^{-\delta t}\ln(c_t)dt + K\cdot\ln(X_T)\right],$$

given the usual wealth dynamics

$$dX_t = X_t\left[u_t^0 r + u_t^1\alpha\right]dt - c_t dt + u^1\sigma X_t dW_t,$$

and the usual control constraints

$$\begin{aligned} c_t &\geq 0,\ \forall t\geq 0, \\ u_t^0 + u_t^1 &= 1,\ \forall t\geq 0. \end{aligned}$$

Exercise 19.2 A **Bernoulli equation** is an ODE of the form

$$\dot{x}_t + A_t x_t + B_t x_t^\alpha = 0,$$

where A and B are deterministic functions of time and α is a constant.

If $\alpha = 1$ this is a linear equation, and can thus easily be solved. Now consider the case $\alpha \neq 1$ and introduce the new variable y by

$$y_t = x_t^{1-\alpha}.$$

Show that y satisfies the **linear** equation

$$\dot{y}_t + (1-\alpha)A_t y_t + (1-\alpha)B_t = 0.$$

Exercise 19.3 Use the previous exercise in order to solve (19.52)–(19.53) explicitly.

Exercise 19.4 The following example is taken from Björk *et al.* (1987). We consider a consumption problem without risky investments, but with stochastic prices for various consumption goods.

$$\begin{aligned} N &= \text{the number of consumption goods}, \\ p_i(t) &= \text{price, at } t\text{, of good } i \text{ (measured as dollars per unit per unit time)}, \\ p(t) &= [p_1(t),\ldots,p_N(t)]', \\ c_i(t) &= \text{rate of consumption of good } i, \\ c(t) &= [c_1(t),\ldots,c_N(t)]', \end{aligned}$$

$$
\begin{aligned}
X(t) &= \text{wealth process}, \\
r &= \text{short rate of interest}, \\
T &= \text{time horizon}.
\end{aligned}
$$

We assume that the consumption price processes satisfy

$$dp_i = \mu_i(p)dt + \sqrt{2}\sigma_i(p)dW_i$$

where $W_1, \ldots, W_n$ are independent. The X-dynamics become

$$dX = rXdt - c'pdt,$$

and the objective is to maximize expected discounted utility, as measured by

$$E\left[\int_0^\tau F(t, c_t)dt\right]$$

where τ is the time of ruin, i.e.

$$\tau = \inf\{t \geq 0; X_t = 0\} \wedge T.$$

(a) Denote the optimal value function by $V(t, x, p)$ and write down the relevant HJB equation (including boundary conditions for $t = T$ and $x = 0$).

(b) Assume that F is of the form

$$F(t, c) = e^{-\delta t} \prod_{i=1}^{N} c_i^{\alpha_i}$$

where $\delta > 0$, $0 < \alpha_i < 1$ and $\alpha = \sum_1^N \alpha_i < 1$. Show that the optimal value function and the optimal control have the structure

$$
\begin{aligned}
V(t, x, p) &= e^{-\delta t} x^\alpha \alpha^{-\alpha} G(t, p), \\
c_i(t, x, p) &= \frac{x}{p_i} \cdot \frac{\alpha_i}{\alpha} A(p)^\gamma G(t, p),
\end{aligned}
$$

where G solves the nonlinear equation

$$
\begin{cases}
\dfrac{\partial G}{\partial t} + (\alpha r - \delta)G + (1 - \alpha)A^\gamma G^{-\alpha\gamma} + \displaystyle\sum_i^N \mu_i \frac{\partial G}{\partial p_i} + \sum_i^N \sigma_i^2 \frac{\partial^2 G}{\partial {p_i}^2} = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G(T, p) = 0, \quad p \in R^N.
\end{cases}
$$

If you find this too hard, then study the simpler case when $N = 1$.

(c) Now assume that the price dynamics are given by GBM, i.e.

$$dp_i = p_i\mu_i dt + \sqrt{2}p_i\sigma_i dW_i.$$

Try to solve the G-equation above by making the **ansatz**

$$G(t,p) = g(t)f(p).$$

Warning: This becomes somwhat messy.

Exercise 19.5 Consider as before state process dynamics

$$dX_t = \mu\left(t, X_t, u_t\right) dt + \sigma\left(t, X_t, u_t\right) dW_t$$

and the usual restrictions for u. Our entire derivation of the HJB equation has so far been based on the fact that the objective function is of the form

$$\int_0^T F(t, X_t, u_t)dt + \Phi(X_T).$$

Sometimes it is natural to consider other criteria, like the expected **exponential utility** criterion

$$E\left[\exp\left\{\int_0^T F(t, X_t, u_t)dt + \Phi(X_T)\right\}\right].$$

For this case we define the optimal value function as the supremum of

$$E_{t,x}\left[\exp\left\{\int_t^T F(s, X_s, u_s)dt + \Phi(X_T)\right\}\right].$$

Follow the reasoning in Section **??** in order to show that the HJB equation for the expected exponential utility criterion is given by

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,x) + \sup_u \left\{V(t,x)F(t,x,u) + \mathcal{A}^u V(t,x)\right\} = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V(T,x) = e^{\Phi(x)}. \end{cases}$$

Exercise 19.6 Solve the problem to minimize

$$E\left[\exp\left\{\int_0^T u_t^2 dt + X_T^2\right\}\right]$$

given the scalar dynamics

$$dX = (ax + u)dt + \sigma dW$$

where the control u is scalar and there are no control constraints.

Hint: Make the ansatz

$$V(t,x) = e^{A(t)x^2 + B(t)}.$$

Exercise 19.7 Study the general linear–exponential–qudratic control problem of minimizing

$$E\left[\exp\left\{\int_0^T \left\{X_t' Q X_t + u_t' R u_t\right\} dt + X_T' H X_T\right\}\right]$$

given the dynamics

$$dX_t = \left\{AX_t + Bu_t\right\} dt + CdW_t.$$

Exercise 19.8 The object of this exercise is to connect optimal control to martingale theory. Consider therefore a general control problem of minimizing

$$E\left[\int_0^T F(t, X_t^{\mathbf{u}}, \mathbf{u}_t)dt + \Phi\left(X_T^{\mathbf{u}}\right)\right]$$

given the dynamics

$$dX_t = \mu\left(t, X_t, u_t\right)dt + \sigma\left(t, X_t, u_t\right)dW_t,$$

and the constraints

$$\mathbf{u}(t, x) \in U.$$

Now, for any control law $\mathbf{u}$, define the **total cost process** $C(t; \mathbf{u})$ by

$$C(t; \mathbf{u}) = \int_0^t F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + E_{t, X_t^{\mathbf{u}}}\left[\int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)dt + \Phi\left(X_T^{\mathbf{u}}\right)\right],$$

i.e.

$$C(t; \mathbf{u}) = \int_0^t F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \mathcal{J}(t, X_t^{\mathbf{u}}, \mathbf{u}).$$

Use the HJB equation in order to prove the following claims.

(a) If $\mathbf{u}$ is an arbitrary control law, then C is a submartingale.

(b) If $\mathbf{u}$ is optimal, then C is a martingale.

20

THE MARTINGALE APPROACH TO OPTIMAL INVESTMENT*

20.1 Exercises

Exercise 20.1 In this exercise we will see how intermediate consumption can be handled by the martingale approach. We make the assumptions of Section 20.1 and the problem is to maximize

$$E^P\left[\int_0^T g(s, c_s)ds + U(X_T)\right]$$

over the class of self financing portfolios with initial wealth x. Here c is the consumption rate for a consumption good with unit price, so c denotes consumption rate in terms of dollars per unit time. The function g is the local utility of consumption, U is utility of terminal wealth, and X is portfolio wealth.

(a) Convince yourself that the appropriate budget constraint is given by

$$E^Q\left[\int_0^T e^{-rs}c_s ds + e^{-rT}X_T\right] = x.$$

(b) Show that the first order condition for the optimal terminal wealth and the optimal consumption plan are given by

$$\begin{aligned}\hat{c}_t &= G(\lambda e^{-rt}L_t),\\ \hat{X}_X &= F(\lambda e^{-rT}L_T),\end{aligned}$$

where $G = (g')^{-1}$, $F = (U')^{-1}$, L is the usual likelihood process, and λ is a Lagrange multiplier.

Exercise 20.2 Consider the setup in the previous exercise and assume that $g(c) = \ln(c)$ and $U(x) = a\ln(x)$, where a is a positive constant. Compute the optimal consumption plan, and the optimal terminal wealth profile.

Exercise 20.3 Consider the log-optimal portfolio given by Proposition 20.9 as

$$X_t = e^{rt}xL_t^{-1}.$$

Show that this portfolio is the "P numeraire portfolio" in the sense that if Π is the arbitrage free price process for any asset in the economy, underlying or derivative, then the normalized asset price

$$\frac{\Pi_t}{X_t}$$

is a martingale under the objective probability measure P.

21

OPTIMAL STOPPING THEORY AND AMERICAN OPTIONS*

21.1 Exercises

Exercise 21.1 Prove that, in discrete time, random time τ is a stopping time if and only if $\{\tau = n\} \in \mathcal{F}_n$ for all n.

Exercise 21.2 Construct a (trivial) example in continuous time, of an optimal stopping problem for which there is no optimal stopping time.

Exercise 21.3 Prove Proposition 21.2.

Exercise 21.4 Prove Proposition 21.3.

Exercise 21.5 Prove Proposition 21.4.

Exercise 21.6 Prove Proposition 21.5.

Exercise 21.7 Let f and g be concave functions. Show that h defined by $h(x) = \min\{f(x), g(x)\}$ is concave.

Exercise 21.8 Let X and Y be supermartingales. Show that Z defined by

$$Z_n(\omega) = \min\{X_n(\omega), Y_n(\omega)\}$$

is a supermartingale. Compare with the previous exercise and with the relations between martingale theory and convex theory given in Section 21.3.

Exercise 21.9 Prove Proposition 21.14.

Exercise 21.10 Assume a standard Black-Scholes model for the stock price and assume that $r = 0$. In this (highly unrealistic) case, one can easily solve the American put problem on a finite time horizon $[0, T]$. Do this.

Exercise 21.11 Consider an ODE of the form

$$f(s) + asf'(s) + bs^2 f''(s) = 0.$$

Introduce a new variable x by $x = \ln(s)$. Show that the ODE by this change of variable will be transformed into a linear ODE with constant coefficients. More precisely, find the ODE satisfied by the function F, defined by $F(x) = f(e^x)$, i.e. $f(s) = F(\ln s)$.

Exercise 21.12 Consider the ODE (21.69). Use the transformation in the previous exercise to show that the ODE has a general solution of the form (21.70).

Exercise 21.13 Prove Proposition 21.30.

22

BONDS AND INTEREST RATES

22.1 Exercises

Exercise 22.1 A **forward rate agreement** (FRA) is a contract, by convention entered into at $t = 0$, where the parties (a lender and a borrower) agree to let a certain interest rate, $R^\star$, act on a prespecified principal, K, over some future period $[S, T]$. Assuming that the interest rate is continuously compounded, the cash flow to the lender is, by definition, given as follows:

At time S: $-K$.

At time T: $Ke^{R^\star(T-S)}$.

The cash flow to the borrower is of course the negative of that to the lender.

(a) Compute for any time $t < S$, the value, $\Pi(t)$, of the cash flow above in terms of zero coupon bond prices.

(b) Show that in order for the value of the FRA to equal zero at $t = 0$, the rate $R^\star$ has to equal the forward rate $R(0; S, T)$ (compare this result to the discussion leading to the definition of forward rates).

Exercise 22.2 Prove the first part of Proposition 22.5.

Hint: Apply the Itô formula to the process $\log p(t, T)$, write this in integrated form and differentiate with respect to T.

Exercise 22.3 Consider a coupon bond, starting at T_0, with face value K, coupon payments at $T_1, \ldots, T_n$ and a fixed coupon rate r. Determine the coupon rate r, such that the price of the bond, at T_0, equals its face value.

Exercise 22.4 Derive the pricing formula (**??**) directly, by constructing a self-financing portfolio which replicates the cash flow of the floating rate bond.

Exercise 22.5 Let $\{y(0, T);\ T \geq 0\}$ denote the zero coupon yield curve at $t = 0$. Assume that, apart from the zero coupon bonds, we also have exactly one fixed coupon bond for every maturity T. We make no particular assumptions about the coupon bonds, apart from the fact that all coupons are positive, and we denote the yield to maturity, again at time $t = 0$, for the coupon bond with maturity T, by $y_M(0, T)$. We now have three curves to consider: the forward rate curve $f(0, T)$, the zero coupon yield curve $y(0, T)$, and the coupon yield curve $y_M(0, T)$. The object of this exercise is to see how these curves are connected.

(a) Show that

$$f(0, T) = y(0, T) + T \cdot \frac{\partial y(0, T)}{\partial T}.$$

(b) Assume that the zero coupon yield cuve is an increasing function of T. Show that this implies the inequalities

$$y_M(0,T) \leq y(0,T) \leq f(0,T), \quad \forall T,$$

(with the opposite inequalities holding if the zero coupon yield curve is decreasing). Give a verbal economic explanation of the inequalities.

Exercise 22.6 Prove Proposition 22.11.

Exercise 22.7 Consider a **consol bond**, i.e. a bond which will forever pay one unit of cash at $t = 1, 2, \ldots$. Suppose that the market yield y is constant for all maturities.

(a) Compute the price, at $t = 0$, of the consol.

(b) Derive a formula (in terms of an infinite series) for the duration of the consol.

(c) Use (a) and Proposition 22.11 in order to compute an analytical formula for the duration.

(d) Compute the convexity of the consol.

23

SHORT RATE MODELS

23.1 Exercises

Exercise 23.1 We take as given an interest rate model with the following P-dynamics for the short rate.

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))d\bar{W}(t).$$

Now consider a T-claim of the form $\mathcal{X} = \Phi(r(T))$ with corresponding price process $\Pi(t)$.

(a) Show that, under any martingale measure Q, the price process $\Pi(t)$ has a local rate of return equal to the short rate of interest. In other words, show that the stochastic differential of $\Pi(t)$ is of the form

$$d\Pi(t) = r(t)\Pi(t)\,dt + \sigma_\Pi \Pi(t)\,dW(t).$$

(b) Show that the normalized price process

$$Z(t) = \frac{\Pi(t)}{B(t)}$$

is a Q-martingale.

Exercise 23.2 The object of this exercise is to connect the forward rates defined in Chapter 22 to the framework above.

(a) Assuming that we are allowed to differentiate under the expectation sign, show that

$$f(t,T) = \frac{E^Q_{t,r(t)}\left[r(T)\exp\left\{-\int_t^T r(s)ds\right\}\right]}{E^Q_{t,r(t)}\left[\exp\left\{-\int_t^T r(s)ds\right\}\right]}.$$

(b) Check that indeed $r(t) = f(t,t)$.

Exercise 23.3. (Swap a fixed rate vs. a short rate) Consider the following version of an interest rate swap. The contract is made between two parties, A and B, and the payments are made as follows.

A (hypothetically) invests the principal amount K at time 0 and lets it grow at a fixed rate of interest R (to be determined below) over the time interval $[0,T]$.

At time T the principal will have grown to K_A SEK. A will then subtract the principal amount and pay the surplus $K - K_A$ to B (at time T).

B (hypothetically) invests the principal at the stochastic short rate of interest over the interval $[0,T]$.

At time T the principal will have grown to K_B SEK. B will then subtract the principal amount and pay the surplus $K - K_B$ to A (at time T).

The **swap rate** for this contract is now defined as the value, R, of the fixed rate which gives this contract the value zero at $t = 0$. Your task is to compute the swap rate.

Exercise 23.4. (Forward contract) Consider a model with a stochastic rate of interest. Fix a T-claim $\mathcal{X}$ of the form $\mathcal{X} = \Phi(r(T))$, and fix a point in time t, where $t < T$. From Proposition 23.4 we can in principle compute the arbitrage free price for $\mathcal{X}$ if we pay at time t. We may also consider a **forward contract** (see Section **??**) on $\mathcal{X}$ **contracted at** t. This contract works as follows, where we assume that you are the buyer of the contract.

At time T you obtain the amount $\mathcal{X}$ SEK.

At time T you pay the amount K SEK.

The amount K is determined at t.

The **forward price for** $\mathcal{X}$ **contracted at** t is defined as the value of K which gives the entire contract the value zero at time t. Give a formula for the forward price.

24

MARTINGALE MODELS FOR THE SHORT RATE

24.1 Exercises

Exercise 24.1 Consider the Vasiček model, where we always assume that $a > 0$.

(a) Solve the Vasiček SDE explicitly, and determine the distribution of $r(t)$.

Hint: The distribution is Gaussian (why?), so it is enough to compute the expected value and the variance.

(b) As $t \to \infty$, the distribution of $r(t)$ tends to a limiting distribution. Show that this is the Gaussian distribution $N[b/a, \sigma/\sqrt{2a}]$. Thus we see that, in the limit, r will indeed oscillate around its mean reversion level b/a.

(c) Now assume that $r(0)$ is a stochastic variable, independent of the Wiener process W, and by definition having the Gaussian distribution obtained in (b). Show that this implies that $r(t)$ has the limit distribution in (b), for all values of t. Thus we have found the stationary distribution for the Vasiček model.

(d) Check that the density function of the limit distribution solves the time invariant Fokker–Planck equation, i.e. the Fokker–Planck equation with the $\frac{\partial}{\partial t}$-term equal to zero.

Exercise 24.2 Show directly that the Vasiček model has an affine term structure without using the methodology of Proposition 24.2. Instead use the characterization of $p(t,T)$ as an expected value, insert the solution of the SDE for r, and look at the structure obtained.

Exercise 24.3 Try to carry out the program outlined above for the Dothan model and convince yourself that you will only get a mess.

Exercise 24.4 Show that for the Dothan model you have $E^Q[B(t)] = \infty$.

Exercise 24.5 Consider the Ho–Lee model

$$dr = \Theta(t)dt + \sigma dW(t).$$

Assume that the observed bond prices at $t = 0$ are given by $\{p^\star(0,T);\ T \geq 0\}$. Assume furthermore that the constant σ is given. Show that this model can be fitted exactly to today's observed bond prices with Θ as

$$\Theta(t) = \frac{\partial f^\star}{\partial T}(0,t) + \sigma^2 t,$$

where $f^\star$ denotes the observed forward rates. (The observed bond price curve is assumed to be smooth.)

Hint: Use the affine term strucuture, and fit forward rates rather than bond prices (this is logically equivalent).

Exercise 24.6 Use the result of the previous exercise in order to derive the bond price formula in Proposition 24.4.

Exercise 24.7 It is often considered reasonable to demand that a forward rate curve always has an horizontal asymptote, i.e. that $\lim_{T\to\infty} f(t,T)$ exists for all t. (The limit will obviously depend upon t and $r(t)$). The object of this exercise is to show that the Ho–Lee model is not consistent with such a demand.

(a) Compute the explicit formula for the forward rate curve $f(t,T)$ for the Ho–Lee model (fitted to the initial term structure).

(b) Now assume that the initial term structure indeed has a horizontal asymptote, i.e. that $\lim_{T\to\infty} f^{\star}(0,T)$ exists. Show that this property is not respected by the Ho–Lee model, by fixing an arbitrary time t, and showing that $f(t,T)$ will be asymptotically linear in T.

Exercise 24.8 The object of this exercise is to indicate why the CIR model is connected to squares of linear diffusions. Let Y be given as the solution to the following SDE.

$$dY = \left(2aY + \sigma^2\right) dt + 2\sigma\sqrt{Y} dW, \quad Y(0) = y_0.$$

Define the process Z by $Z(t) = \sqrt{Y(t)}$. It turns out that Z satisfies a stochastic differential equation. Which?

25

FORWARD RATE MODELS

25.1 Exercises

Exercise 25.1 Show that for the Hull–White model

$$dr = (\Theta(t) - ar)dt + \sigma dW$$

the corresponding HJM formulation is given by

$$df(t,T) = \alpha(t,T)dt + \sigma e^{-a(T-t)}dW.$$

Exercise 25.2 (**Gaussian interest rates**) Take as given an HJM model (under the risk neutral measure Q) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where the volatility $\sigma(t,T)$ is a **deterministic** function of t and T.

(a) Show that all forward rates, as well as the short rate, are normally distributed.

(b) Show that bond prices are log-normally distributed.

Exercise 25.3 Consider the domestic and a foreign bond market, with bond prices being denoted by $p_d(t,T)$ and $p_f(t,T)$ respectively. Take as given a standard HJM model for the domestic forward rates $f_d(t,T)$, of the form

$$df_d(t,T) = \alpha_d(t,T)dt + \sigma_d(t,T)dW(t),$$

where W is a multidimensional Wiener process under the **domestic** martingale measure Q. The foreign forward rates are denoted by $f_f(t,T)$, and their dynamics, still under the domestic martingale measure Q, are assumed to be given by

$$df_f(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW(t).$$

Note that the same vector Wiener process is driving both the domestic and the foreign bond market. The exchange rate X (denoted in units of domestic currency per unit of foreign currency) has the Q dynamics

$$dX(t) = \mu(t)X(t)dt + X(t)\sigma_X(t)dW(t).$$

Under a foreign martingale measure, the coefficient processes for the foreign forward rates will of course satisfy a standard HJM drift condition, but here we have given the dynamics of f_f under the domestic martingale measure Q. Show that under this measure the foreign forward rates satisfy the modified drift condition

$$\alpha_f(t,T) = \sigma_f(t,T)\left\{\int_t^T \sigma_f'(t,s)ds - \sigma_X'(t)\right\}.$$

Exercise 25.4 With notation as in the exercise above, we define the **yield spread** $g(t,T)$ by

$$g(t,T) = f_f(t,T) - f_d(t,T).$$

Assume that you are given the dynamics for the exchange rate and the domestic forward rates as above. You are also given the spread dynamics (again under the domestic measure Q) as

$$dg(t,T) = \alpha_g(t,T)dt + \sigma_g(t,T)dW(t).$$

Derive the appropriate drift condition for the coefficient process α_g in terms of σ_g, σ_d and σ_X (but not involving σ_f).

Exercise 25.5 A **consol bond** is a bond which forever pays a constant continuous coupon. We normalize the coupon to unity, so over every interval with lenght dt the consol pays $1 \cdot dt$. No face value is ever paid. The price $C(t)$, at time t, of the consol is the value of this infinite stream of income, and it is obviously (why?) given by

$$C(t) = \int_t^\infty p(t,s)ds.$$

Now assume that bond price dynamics under a martingale measure Q are given by

$$dp(t,T) = p(t,T)r(t)dt + p(t,T)v(t,T)dW(t),$$

where W is a vector valued Q-Wiener process. Use the heuristic arguments given in the derivation of the HJM drift condition (see Section 22.2.2) in order to show that the consol dynamics are of the form

$$dC(t) = (C(t)r(t) - 1)\, dt + \sigma_C(t)dW(t),$$

where

$$\sigma_C(t) = \int_t^\infty p(t,s)v(t,s)ds.$$

26

CHANGE OF NUMERAIRE*

26.1 Exercises

Exercise 26.1 Derive a pricing formula for European bond options in the Ho–Lee model.

Exercise 26.2 A Gaussian Interest Rate Model
Take as given a HJM model (under the risk neutral measure Q) of the form

$$df(t,T) = \alpha(t,T)dt + \sigma_1 \cdot (T-t)dW_1(t) + \sigma_2 e^{-a(T-t)}dW_2(t)$$

where σ_1 and σ_2 are constants.

(a) Derive the bond price dynamics.

(b) Compute the pricing formula for a European call option on an underlying bond.

Exercise 26.3 Prove that a payment of $\frac{1}{p}(A-p)^+$ at time T_i is equivalent to a payment of $(A-p)^+$ at time T_{i-1}, where $p = p(T_{i-1}, T_i)$, and A is a deterministic constant.

Exercise 26.4 Prove Lemma 26.9.

Exercise 26.5 Use the technique above in order to prove the pricing formula of Proposition 24.5 for bond options in the Ho–Lee model.

27

LIBOR AND SWAP MARKET MODELS

27.1 Exercises

Exercise 27.1 Prove that the arbitrage free value at $t \le T_n$ of the T_{i+1} claim

$$\alpha_{i+1} \cdot L_{i+1}(T_i)$$

is given by

$$p(t, T_i) - p(t, T_{i+1}).$$

Exercise 27.2 Convince yourself that the swap measure Q_{N-1}^N equals the forward measure Q^{T_N}.

Exercise 27.3 Show that the arbitrage free price for a payer swap with swap rate K is given by the formula

$$\mathbf{PS_n^N}(t; K) = \left(R_n^N(t) - K\right) S_n^N(t).$$

Exercise 27.4 Prove Proposition 27.17.

28

POTENTIALS AND POSITIVE INTEREST

28.1 Exercises

Exercise 28.1 Prove that the term structure constructed in Proposition 28.5 is positive.

Exercise 28.2 Prove proposition 28.6.

Exercise 28.3 Prove the first part of Proposition 28.8.

Exercise 28.4 Assume that the process Y has a stochastic differential of the form

$$dY_t = \{\alpha Y_t + \beta_t\}\, dt + \sigma_t dW_t,$$

where α is a real number whereas β and σ are adapted processes. Show that Y can be written as

$$Y_t = e^{\alpha t} Y_0 + \int_0^t e^{\alpha(t-s)} \beta_s ds + \int_0^t e^{\alpha(t-s)} \sigma_s dW_s.$$

Exercise 28.5 Without the normalizing function Φ, we can write the forward rates in a Flesaker-Hughston model as

$$f(t,T) = \frac{M(t,T)}{\int_T^\infty M(t,s)ds}.$$

A natural way of modeling the positive martingale family $M(t,T)$ is to write

$$dM(t,T) = M(t,T)\sigma(t,T)dW_t$$

for some chosen volatility structure σ, where σ and W are d-dimensional. Show that in this framework the forward rate dynamics are given by

$$df(t,T) = f(t,T)\{v(t,T) - \sigma(t,T)\} v^\star(t,T)dt + f(t,T)\{\sigma(t,T) - v(t,T)\} dW_t,$$

where * denotes transpose and

$$v(t,T) = \frac{\int_T^\infty M(t,s)\sigma(t,s)ds}{\int_T^\infty M(t,s)ds}$$

Exercise 28.6 This exercise describes another way of producing a potential. Consider a fixed random variable $X_\infty \in L^2(P, \mathcal{F}_\infty)$. We can then define a martingale X by setting

$$X_t = E^P\left[X_\infty \middle| \mathcal{F}_t \right].$$

Now define the process Z by

$$Z_t = E^P\left[(X_\infty - X_t)^2 \middle| \mathcal{F}_t \right]$$

(a) Show that

$$Z_t = E^P \left[X_\infty^2 \middle| \mathcal{F}_t \right] - X_t^2.$$

(b) Show that Z is a supermartingale and that Z is, in fact, a potential.

The point of this is that the potential Z, and thus the complete interest rate model generated by Z, is fully determined by a specification of the single random variable X_∞. This is called a "conditional variance model". See the Notes.

29

FORWARDS AND FUTURES

29.1 Exercises

Exercise 29.1 Suppose that S is the price process of a non dividend paying asset. Show that the forward price $f(t, x; T, \mathcal{Y})$ for the T-claim $\mathcal{Y} = S_T$ is given by

$$f(t, x; T, S_T) = \frac{S_t}{p(t, T)}.$$

Exercise 29.2 Suppose that S is the price process of a dividend paying asset with dividend process D.

(a) Show that the forward price $f(t, x; T, S_T)$ is given by the cost of carry formula

$$f(t, x; T, S_T) = \frac{1}{p(t, T)} \left(S_t - E^Q_{t,x} \left[\int_t^T \exp\left\{ - \int_t^s r(u)du \right\} dD_s \right] \right).$$

Hint: Use the cost of carry formula for dividend paying assets.

(b) Now assume that the short rate r is deterministic but possibly time-varying. Show that in this case the formula above can be written as

$$f(t, x; T, S_T) = \frac{S_t}{p(t, T)} - E^Q_{t,x} \left[\int_t^T \exp\left\{ - \int_s^T r(u)du \right\} dD_s \right].$$

Exercise 29.3 Suppose that S is the price process of an asset in a standard Black–Scholes model, with r as the constant rate of interest, and fix a contingent T-claim $\Phi(S(T))$. We know that this claim can be replicated by a portfolio based on the money account B, and on the underlying asset S. Show that it is also possible to find a replicating portfolio, based on the money account and on futures contracts for $S(T)$.