Lévy Processes and Applications

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Chapter 1

Introduction

1.1 Outline

The aim of this lecture notes is to provide students with some mathematical methods related with the theory of Lévy processes and applications in finance. More precisely, we want to give a critical presentation and discussion of the Black-Scholes model imperfections and the possible advantages of Lévy processes in financial modeling, present and discuss the main concepts and results of the Lévy processes theory and its associated stochastic calculus, present and discuss some important applications of Lévy processes in finance. These notes were prepared for the course "Lévy Processes and applications", of the Msc. degree in Financial Mathematics in ISEG, Technical University of Lisbon, in the academic year 2012/2013. In some parts of the text, we will follow the references [1], [2], [5] and [7].

1.2 Main concepts, applications and some history

A Lévy process is, basically, a stochastic process that has stationary and independent increments. The theory was developed by Paul Lévy (1886-1971) on the 1930s. Many interesting examples of stochastic processes belong to the class of Lévy processes: Brownian motion, Poisson processes, jump-diffusion processes, subordinated processes, etc. Lévy processes forma a simple class of processes with continuous paths mixed with jumps of random amplitude size. Moreover, Lévy processes are a subset of semimartingales. An important subset of Markov processes are the solutions of stochastic differential equations driven by Lévy random noise. Lévy processes have a robust and stable structure. The applications of Lévy processes usually only require them to take values in Euclidean space. However, this space can be replaced by infinite dimensional functional spaces, like a Hilbert space or even a Banach space.

The main applications of Lévy processes are in the areas of turbulence, finance, Physics and quantum groups. The main areas in finance are option pricing in incomplete markets, interest rate modelling and credit risk modelling.

In 1900, Louis Bachelier used the Brownian motion in order to model the stock prices evolution. This was the first modern model of mathematical finance and describes the stock price evolution of an asset by the stochastic process

$$S_t = S_0 + \sigma B_t,$$

where $S_0 > 0$ is the initial stock price, σ is a real constant and the process $B = \{B_t, t \in [0,T]\}$ is a Brownian motion. This model has the unfortunate disadvantage of allowing negative prices for the stock price. In order to remedy this problem, Samuelson introduced the geometric Brownian motion as follows

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right],$$

where μ and σ are constants. This model can also be introduced as the solution of the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}B_t.$$

In the Samuelson model, the stock price process S_t has a lognormal distribution. That is, $\log S_t - \log S_0$ has the Gaussian distribution $N\left(\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$.

In 1973, Black, Scholes and Merton developed a model known in the literature as the Black-Scholes model, which has revolutionized the finance world and triggered the growth of the derivatives markets. In the Black-Scholes model, investors trade continuously in time. A frictionless market with two assets is assumed. The first asset is a risky asset, which is usually called the stock, with price process $S = \{S_t, t \in [0,T]\}$ given by the geometric Brownian motion. The second asset is a risk-free asset, usually called a bond, and with price process $B = \{B_t, t \in [0,T]\}$ given by

$$B_t = \exp(rt),$$

where r is the constant risk-free rate. The Black-Scholes model is a complete model, in the sense that any contingent claim admits a replicating portfolio. Therefore, there is a unique price for each contingent claim.

The geometric Brownian motion model for stock prices has been very popular. However, it has some important drawbacks. Indeed, the log-returns of stock prices in financial data do not follow a Gaussian distribution (see [4] and [7]). The empirical negative skewness and excess kurtosis of stocks log-returns are very different from the normal distribution skewness and kurtosis. The excess kurtosis in the log-returns of financial data means that large movements in asset prices are much more frequent than in a model with Gaussian log-returns. The empirical distribution has "fat tails", when compared with the Gaussian distribution.

In order to price and hedge contingent claims, it is important to have a good model for the dynamics of the underlying risky assets. This can be achieved by considering probability distributions which are more flexible than the Gaussian distribution, and stochastic processes that generalize the standard Brownian motion, leading to a more precise dynamic modeling of the stock price process. The natural generalizations of Brownian motion, which preserve the important properties of independent and stationary increments but with more general distributions than the Gaussian distribution, are Lévy processes (or non-homogeneous Lévy processes, if we drop the stationarity assumption). These processes are characterized by infinite divisible distributions. In general, the Lévy processes have jumps, which is an important property in order to model the real discontinuous evolution of prices and the tail behavior of the returns distribution.

By replacing the standard Brownian motion in the Black Scholes model with a Lévy process (or a non-homogeneous Lévy process), we define a Lévy market model consisting of a risky asset or stock and a riskless asset or bond. In this model, we consider that the risky asset has a price process $S = \{S_t, t \in [0, T]\}$ given by the so-called geometric Lévy process, which is the solution of the stochastic differential equation

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mathrm{d}Z_t, \qquad S_0 > 0, \tag{1.1}$$

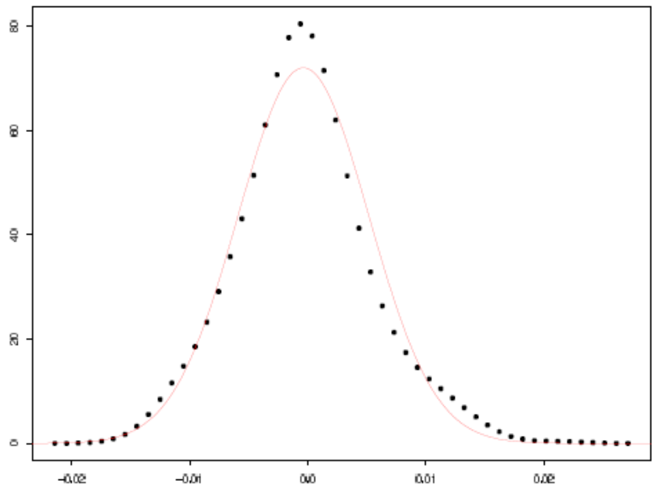
where $Z = \{Z_t, t \in [0, T]\}$ is a Lévy process or a non homogeneous Lévy process.

In recent years, the models based on Lévy processes have become increasingly popular. Comprehensive presentations of the theory of Lévy processes are given in [1], [4] and [6]. Comprehensive reviews of the financial applications are given in [4], [5] and [7].

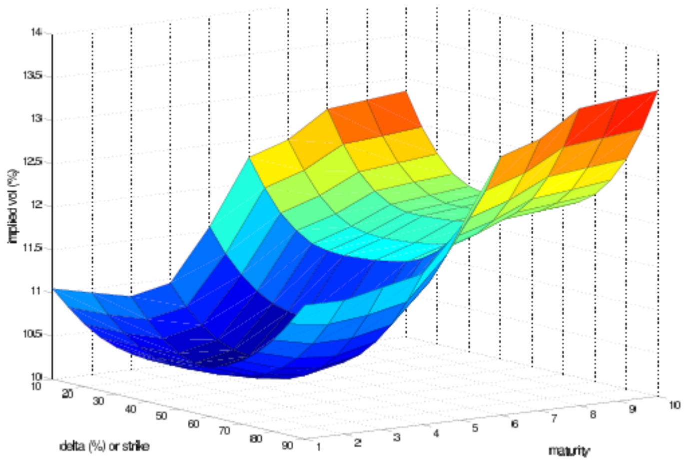
1.3 The Imperfections of the Black-Scholes model

Emprirical data from financial markets shows that (see [5])

- asset price processes have jumps
- Empirical distribution of asset returns has fat tails and skewness.
- Implied volatilities are not constant across strike or maturities.



Empirical Distribution of daily log-returns (GBP/USD exchange rate) and fitted Gaussian distribution (from [5])



Implied volatilities of vanilla options on the EUR/USD exchange rate on November 5, 2001 (from [5])

All this empirical evidence contradicts assumptions of the Black-Scholes model.

1.4 Basic Definitions

Definition 1.1 *A càdlág, adapted, stochastic process $L = \{L_t, t \in [0,T]\}$ is a Lévy process if: (1) $L_0 = 0$ a.s.; (2) L has independent increments; (3) L has stationary increments; and (4) L is stochastically continuous, i.e., for every $t \in [0,T]$ and $\varepsilon > 0$, we have*

$$\lim_{s \to t} \mathbb{P}\left[|L_t - L_s| > \varepsilon\right] = 0.$$

Example 1.2 *An example of a Lévy Process is the compensated jump-diffusion process*

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} Z_k - t\lambda m, \tag{1.2}$$

where N is a Poisson process with parameter λ, the process W_t is a Brownian motion, the sequence of random variables $Z = (Z_k)_{k \geq 1}$ is a i.i.d. sequence with probability distribution F, where $\mathbb{E}[Z_k] = m$.

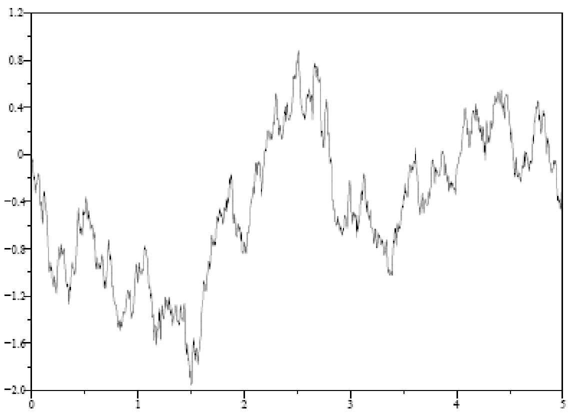


Figure 1 Simulation of standard Brownian motion

A trajectory of a Brownian motion (from [2]).

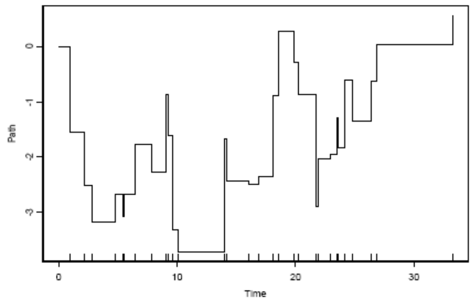
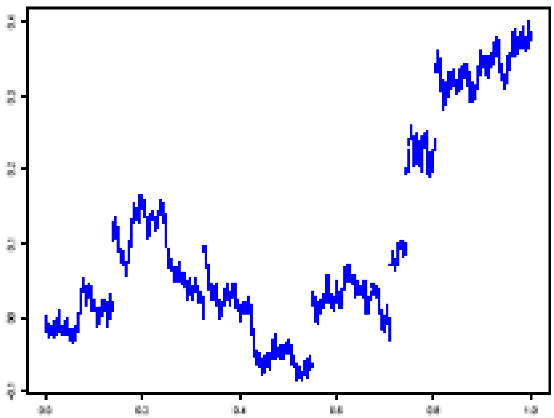


Figure 3. Simulation of a compound Poisson process with $N(0,1)$ summands($\lambda = 1$).

A trajectory of a compound Poisson process (from [2])



A jump-diffusion trajectory (from [5])

The characteristic function of the jump-diffusion process (1.2) is

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left[t\left(iub - \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\right)\lambda F\left(dx\right)\right)\right]. \qquad (1.3)$$

Indeed, we have

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left[iubt\right]\mathbb{E}\left[\exp\left[iu\sigma W_t\right]\right]\mathbb{E}\left[\exp\left[iu\sum_{k=1}^{N_t} Z_k - iut\lambda m\right]\right].$$

and

$$\mathbb{E}\left[\exp\left[iu\sigma W_t\right]\right] = \exp\left[-\frac{1}{2}\sigma^2 u^2 t\right], \qquad W_t \sim N\left(0,t\right),$$

$$\mathbb{E}\left[\exp\left[iu\sum_{k=1}^{N_t} Z_k\right]\right] = \exp\left[\lambda t \mathbb{E}\left[e^{iu\,J} - 1\right]\right], \qquad N_t \sim Po\left(\lambda t\right).$$

Therefore

$$\mathbb{E}\left[e^{iuL_t}\right] = \exp\left[iubt - \frac{\sigma^2 u^2 t}{2}\right] \exp\left[\lambda t \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) \lambda F\,(dx)\right].$$

Definition 1.3 *The law P_X of a random variable X is infinitely divisible if, for all $n \in \mathbb{N}$, exist i.i.d. random variables $X_1^{(1/n)}, X_2^{(1/n)}, \ldots, X_n^{(1/n)}$, such that*

$$X \overset{d}{=} X_1^{(1/n)} + X_2^{(1/n)} + \ldots + X_n^{(1/n)}.$$

The law P_X is infinitely divisible if and only if, for all $n \in \mathbb{N}$, exists a random variable $X^{(1/n)}$ such that

$$\varphi_X(u) = \left(\varphi_{X^{(1/n)}}(u)\right)^n,$$

where $\varphi_X(u)$ is the characteristic function of X and $\varphi_{X^{(1/n)}}$ is the characteristic function of $X^{(1/n)}$.

Example 1.4 *(The Poisson Distribution): Let $X \sim Po(\lambda)$, where $X^{(1/n)} \sim Po\left(\frac{\lambda}{n}\right)$. Then, the characteristic function is*

$$\begin{aligned}\varphi_X(u) &= \exp\left(\lambda\left(e^{iu} - 1\right)\right) \\ &= \left(\exp\left[\frac{\lambda}{n}\left(e^{iu} - 1\right)\right]\right)^n = \left(\varphi_{X^{(1/n)}}(u)\right)^n.\end{aligned}$$

1.5 The Lévy-Kintchine formula

Theorem 1.5 *(Lévy Khintchine formula): The law P_X is infinitely divisible if and only if exists a triplet (b, c, ν), $b \in \mathbb{R}$, $c \geq 0$, where ν is a measure, $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} (1 \wedge x^2)\,\nu(dx) < \infty$ and*

$$\mathbb{E}\left[e^{iuX}\right] = \exp\left[ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)\right].$$

The triplet (b, c, ν) is called the Lévy or characteristic triplet and the exponent

$$\psi(u) = ibu - \frac{u^2 c}{2} + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right) \nu(dx)$$

is called the Lévy or characteristic exponent. b is the drift term, c is the Gaussian or diffusion coefficient and ν is the Lévy measure.

Example 1.6 *The random variable L_t of the jump diffusion process (1.2) has infinitely divisible distribution and $b = bt$, $c = \sigma^2 t$ and $\nu = (\lambda F)\, t$.*

Consider a general Lévy process $L = \{L_t, t \in [0,T]\}$. Then

$$L_t = L_{\frac{t}{n}} + \left(L_{\frac{2t}{n}} - L_{\frac{t}{n}}\right) + \cdots + \left(L_t - L_{\frac{(n-1)t}{n}}\right).$$

By the stationarity and independence of increments, $\left(L_{\frac{kt}{n}} - L_{\frac{(k-1)t}{n}}\right)$ is an i.i.d. sequence. Therefore, L_t has an infinitely divisible distribution.

The characteristic function of a Lévy process is given by the Lévy-Khintchine formula

$$\begin{aligned}\phi_u(t) &= \mathbb{E}\left[e^{iuL_t}\right] = \exp\{t\psi(u)\} \\ &= \exp\left\{t\left(ibu - \frac{u^2 c}{2} + \int_{-\infty}^{+\infty}\left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|<1\}}\right)\nu(dx)\right)\right\},\end{aligned}$$

where ν is the Lévy measure, (b, c, ν) is the triplet of characteristics of the Lévy process and $\psi(u)$ is the characteristic exponent of L_1.

- Every Lévy process can be associated with a infinitely divisible distribution.
- The opposite (Lévy-Itô decomposition) is also true. Given a random variable X with infinitely divisible distribution, we can construct a Lévy process $L = \{L_t, t \in [0,T]\}$ such that the law of L_1 is the law of X.

1.6 Jumps of a Lévy process

The jump process of a Lévy process L is defined as the process $\Delta L = \{\Delta L_t, t \in [0,T]\}$, where

$$\Delta L_t = L_t - L_{t-}.$$

By the stochastic continuity of L, for a fixed t, $\Delta L_t = 0$ a.s. It is possible that

$$\sum_{s\le t} |\Delta L_s| = \infty \quad \text{a.s.}$$

However,

$$\sum_{s\le t} |\Delta L_s|^2 < \infty \quad \text{a.s.}$$

Let $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ such that $0 \notin \overline{A}$. The Poisson random measure associated to the jumps is the counting measure

$$\mu^L(\omega, t, A) = \#\{0 \le s \le t; \Delta L_s \in A\} = \sum_{s \le t} \mathbf{1}_A(\Delta L_s(\omega)).$$

Then, we have that

- $\mu^L(\cdot, A)$ has independent and stationary increments.
- Hence, $\mu^L(\cdot, A)$ is a Poisson process and μ^L is the Poisson random measure.
- The measure ν defined on $\mathcal{B}(\mathbb{R} \setminus \{0\})$ by

  $$\nu(A) = \mathbb{E}\left[\mu^L(1, A)\right] = \mathbb{E}\left[\sum_{s \le 1} \mathbf{1}_A(\Delta L_s(\omega))\right]$$

  is the Lévy measure of the Lévy process L.

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function on A. The integral of f with respect to the Poisson random measure is defined by

$$\int_A f(x)\, \mu^L(\omega, t, dx) = \sum_{s \le t} f(\Delta L_s)\, \mathbf{1}_A(\Delta L_s(\omega)).$$

Each $\int_A f(x)\, \mu^L(t, dx)$ is a random variable and $\int_0^t \int_A f(x)\, \mu^L(ds, dx)$ is a stochastic process.

Theorem 1.7 *The process $\int_0^t \int_A f(x)\, \mu^L(ds, dx)$ is a compound Poisson process with characteristic function*

$$\exp\left(t \int_A \left(e^{iuf(x)} - 1\right) \nu(dx)\right).$$

If $f \in L^1(A)$ then

$$\mathbb{E}\left[\int_0^t \int_A f(x)\, \mu^L(ds, dx)\right] = t \int_A f(x)\, \nu(dx).$$

If $f \in L^2(A)$ then

$$\mathrm{Var}\left(\left|\int_0^t \int_A f(x)\, \mu^L(ds, dx)\right|\right) = t \int_A |f(x)|^2\, \nu(dx).$$

1.7 The Lévy-Itô decomposition

Theorem 1.8 *Consider the characteristic triple* (b, c, ν) *of an infinitely divisible law. Then, exists a probability space and four Lévy processes* $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ *and* $L^{(4)}$ *(independent) such that*

$$L = L^{(1)} + L^{(2)} + L^{(3)} + L^{(4)}$$

is a Lévy process with triplet (b, c, ν) *and*

$$L_t^{(1)} = bt; \quad L_t^{(2)} = \sqrt{c}W_t,$$
$$L_t^{(3)} = \int_0^t \int_{|x|\geq 1} x\mu^L (ds, dx),$$
$$L_t^{(4)} = \int_0^t \int_{|x|<1} x \left(\mu^L - \nu^L\right)(ds, dx).$$

The Lévy measure, paths and moment properties:

- ν satisfies $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} (1 \wedge x^2)\, \nu(dx) < \infty$ and gives the expected number of jumps of a certain size per unit of time.
- If $\nu(\{\mathbb{R}\}) = \infty$ then infinitely many jumps occur (small jumps). The Lévy process has infinite activity.
- If $\nu(\{\mathbb{R}\}) < \infty$ then a.a. paths have a finite number of jumps. The Lévy process has finite activity.
- Let L be a Lévy process with characteristic triplet (b, c, ν). If $c = 0$ and $\int_{|x|\leq 1} |x|\, \nu(dx) < \infty$ then a.a. paths have finite variation. If $c \neq 0$ or $\int_{|x|\leq 1} |x|\, \nu(dx) = \infty$ then a.a. paths have infinite variation.

The path variation properties are related with small jumps and Brownian motion. The activity depends of all the jumps. The moment properties depend only of the big jumps. The moments of a Lévy processes are finite if and only if certain integrals over the Lévy measure (considering only big jumps) are finite.

- L_t has finite moment of order p if and only if
$$\int_{|x|\geq 1} |x|^p\, \nu(dx) < \infty.$$
- L_t has finite exponential moment of order p $\left(\text{i.e. } \mathbb{E}\left[e^{pL_t}\right] < \infty\right)$ if and only if
$$\int_{|x|\geq 1} e^{px} \nu(dx) < \infty.$$

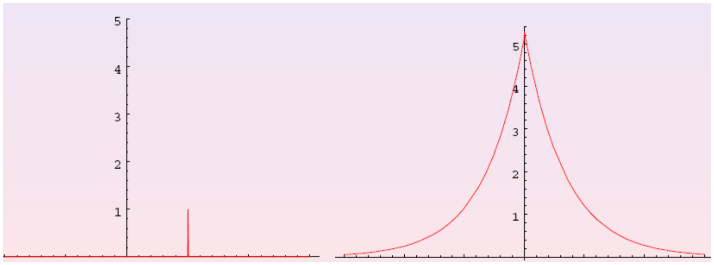


Figure 1.1: The Lévy measure of the Poisson and of a compound Poisson process (from [5])

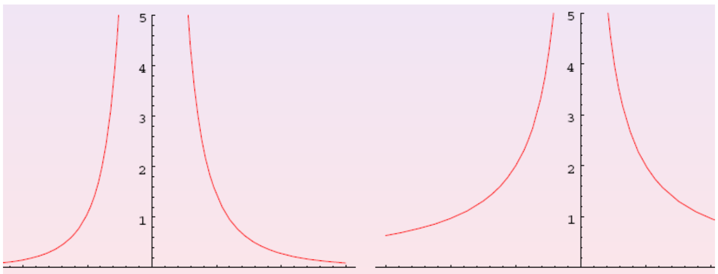


Figure 1.2: The Lévy measure of a NIG and an α-stable process (from [5])

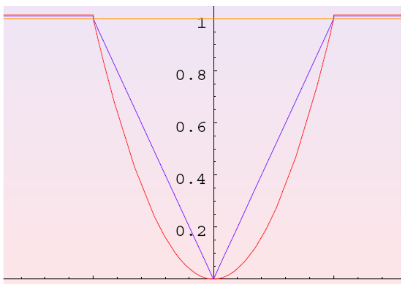


Figure 1.3: $|x|^2 \wedge 1$ (red). $|x| \wedge 1$ (blue) (from [5])

1.8 Some models

A subordinator is an a.s. increasing (in t) Lévy process. A Lévy process is a subordinator if $\nu(-\infty, 0) = 0$, $c = 0$, $\int_{(0,1)} x\nu(dx) < \infty$ and $b \geq 0$. The characteristic exponent of a subordinator is

$$\psi(u) = ibu + \int_0^\infty \left(e^{iux} - 1\right) \nu(dx)$$

The Poisson process is clearly a subordinator.

In the risk neutral-world, the asset price process can be given by an exponential Lévy process:

$$S_t = S_0 \exp(L_t), \qquad 0 \leq t \leq T$$

where L_t is a Lévy process with triplet $\left(\overline{b}, \overline{c}, \overline{\nu}\right)$.and canonical decomposition

$$L_t = \overline{b}t + \sqrt{\overline{c}}\overline{W}_t + \int_0^t \int_{\mathbb{R}} x\left(\mu^L - \overline{\nu}^L\right)(ds, dx)$$

with

$$\overline{b} = r - q - \frac{\overline{c}}{2} - \int_{\mathbb{R}} (e^x - 1 - x)\, \overline{\nu}(dx)$$

- Black-Scholes model: $L_1 \sim N(\mu, \sigma^2)$. The Lévy triplet is $(\mu, \sigma^2, 0)$ and $L_t = \mu t + \sigma W_t$.

- Merton (jump-diffusion) model: $L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k$, with $J_k \sim N(\mu_J, \sigma_J^2)$ (with density f_J). The Lévy triplet is $(\mu, \sigma^2, \lambda \times f_J)$.

- Generalized Hyperbolic model: $L_1 \sim GH(\alpha, \beta, \delta, \mu, \lambda)$ and $L_t = t\mathbb{E}[L_1] + \int_0^t \int_{\mathbb{R}} x\left(\mu^L - \nu^{GH}\right)(ds, dx)$. Lévy triplet: $\left(\mathbb{E}[L_1], 0, \nu^{GH}\right)$. The parameters are such that $\alpha > 0$ is related to the form or shape; $0 \leq |\beta| < \alpha$ is related to the skewness; μ is a location parameter; $\delta > 0$ is a scaling parameter and λ is related with "fat tails".

- The Variance Gamma process: It has a Variance Gamma distribution $VG(\sigma, \nu, \theta)$ with characteristic function

  $$\varphi_u(t) = \left(1 - iu\theta v + \frac{1}{2}\sigma^2 \nu u^2\right)^{-\frac{t}{\nu}}.$$

  It has Lévy triplet $(0, 0, \nu_{VG}(dx))$. The Variance Gamma process can be defined as a time-transformed Brownian motion with drift:

  $$L_t = \theta G_t + \sigma W_{G_t},$$

  where G is a Gamma process with two appropriate parameters.

Chapter 2

Lévy Processes

2.1 Infinitely divisible distributions

Consider a probability space $(\Omega, \mathcal{F}, P)$. The law of the random variable X is given by

$$p_X(A) = P(X \in A),$$

for $A \in \mathcal{F}$. If Y and Z are independent random variables then the law of $Y + Z$ is given by the convolution of laws:

$$p_{Y+Z} = p_Y * p_Z,$$

which is defined by

$$(p_Y * p_Z)(A) = \int_{\mathbb{R}^d} p_Y(A - y)\, p_Z(dy)$$

Equivalently

$$\int_{\mathbb{R}^d} g(z)\,(p_Y * p_Z)(dz) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(y+z)\, p_Y(dy)\, p_Z(dz)$$

for all bounded measurable functions g. If Y and Z are independent, with densities given by f_Y and f_Z, then

$$f_{Y+Z}(y) = (f_Y * f_Z)(y) = \int_{\mathbb{R}^d} f_Y(y - z)\, f_Z(z)\, dz.$$

The characteristic function of a random variable X is $\varphi_X:\mathbb{R}^d \to \mathbb{C}$, defined by

$$\varphi_X(u) = E\left[e^{i(u,X)}\right] = \int_{\mathbb{R}^d} e^{i(u,x)} p(dx).$$

Consider a probability measure on $\mathbb{R}^d$ given by μ. Then, the characteristic function of μ satisfies:

1. $\varphi_\mu(0) = 1$.
2. φ_μ is positive definite: $\sum_{i,j} a_i \overline{a_j} \varphi_\mu(u_i - u_j) \geq 0$ for all $a_i \in \mathbb{C}$, $u_i \in \mathbb{R}^d$, $1 \leq i, j \leq n$, $n \in \mathbb{N}$.
3. $|\varphi_\mu(u)| \leq 1$.
4. φ_μ is uniformly continuous.

- Moreover, $\mu \to \varphi_\mu$ is injective
- Bochner Theorem: If φ:$\mathbb{R}^d \to \mathbb{C}$ satisfies 1. and 2. and is continuous at $u = 0$, then φ is a characteristic function.

Exercise 2.1 *Show that* $|\varphi_\mu(u)| \leq 1$.

Consider the notation: $\mu^{*n} = \mu * \mu * \cdots * \mu$

Definition 2.2 *μ has a convolution n-th root if there is a probability measure $\mu^{\frac{1}{n}}$ such that*

$$\left(\mu^{\frac{1}{n}}\right)^{*n} = \mu.$$

Definition 2.3 *μ is infinitely divisible if exists a convolution n-th root for each $n \in \mathbb{N}$.*

Theorem 2.4 *μ is infinitely divisible if and only if for each $n \in \mathbb{N}$, exists μ_n with characteristic function φ_n:*

$$\varphi_\mu(u) = (\varphi_n(u))^n$$

for all $u \in \mathbb{R}^d$.

Proof. ($\Longrightarrow$) Take $\varphi_n = \varphi_{\mu^{1/n}}$ ■

($\Longleftarrow$) For each n, by Fubini's theorem:

$$\begin{aligned} \varphi_\mu(u) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{i(u, y_1 + y_2 + \cdots + y_n)} \mu_n(dy_1) \cdots \mu_n(dy_n) \\ &= \int_{\mathbb{R}^d} e^{i(u,y)} (\mu_n)^{*n}(dy) \end{aligned}$$

and since φ determines μ uniquely, then $\mu = (\mu_n)^{*n}$ and is infinitely divisible.

Proposition 2.5 *Properties:*

1. *If μ and ν are infinitely divisible distributions then $\mu * \nu$ is infinitely divisible.*

2. *If $\{\mu_n, n \in \mathbb{N}\}$ is a sequence of infinitely divisible distributions and $\mu_n \xrightarrow{w} \mu$, then μ is infinitely divisible.*

Remark 2.6 *$\mu_n \xrightarrow{w} \mu$ means that μ_n converges weakly to μ, i.e.,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\,\mu_n(dx) = \int_{\mathbb{R}^d} f(x)\,\mu(dx)$$

for all $f \in C_b(\mathbb{R}^d)$ (bounded continuous functions).

Exercise 2.7 *Show that Property 1 holds*

Definition 2.8 *A random variable X is infinitely divisible if the law p_X is infinitely divisible. This means that*

$$X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)},$$

where $Y_1^{(n)}, \ldots, Y_n^{(n)}$ are i.i.d., for each $n \in \mathbb{N}$.

Proposition 2.9 *The following statements are equivalent:*

1. *X is infinitely divisible.*

2. *μ_X has a convolution n-th root which is the law of a random variable, for each $n \in \mathbb{N}$*

3. *φ_X has an n-th root which is the characteristic function of some random variable, for each $n \in \mathbb{N}$*

Exercise 2.10 *Prove the previous Proposition.*

Exercise 2.11 *Let $\alpha > 0$, $\beta > 0$. Show that the gamma-(α, β) distribution*

$$\mu_{\alpha,\beta}(dx) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx, \quad \text{with } x > 0$$

is an infinitely-divisible distribution.

In each of the following examples, we will find iid $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$.

Example 2.12 *(Gaussian random variable) Let* $X = (X_1, X_2, \ldots, X_d)$ *be Gaussian random vector, with density:*

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{\det(A)}} \exp\left(-\frac{1}{2}\left(x-m, A^{-1}(x-m)\right)\right), \qquad x \in \mathbb{R}^d.$$

Then $X \sim N(m, A)$, *where* A *is a* $d \times d$ *matrix which is positive-definite and symmetric. It is the covariance matrix:* $A = E\left[(X-m)(X-m)^T\right]$. *We can show that*

$$\varphi_X(u) = \exp\left(i(m,u) - \frac{1}{2}(u, Au)\right).$$

Therefore:

$$(\varphi_X(u))^{\frac{1}{n}} = \exp\left(i\left(\frac{m}{n}, u\right) - \frac{1}{2}\left(u, \frac{1}{n}Au\right)\right).$$

and X *is infinitely divisible with* $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ *and*

$$Y_j^{(n)} \sim N\left(\frac{m}{n}, \frac{1}{n}A\right).$$

Example 2.13 *(Poisson random variable) Let* $d = 1$ *and* $X : \Omega \to \mathbb{N}_0$ *with* $X \sim Po(\lambda)$, *i.e.*

$$P(X = n) = \frac{\lambda^n}{n!}e^{-\lambda}.$$

It is well known that $E[X] = Var[X] = \lambda$ *and it is easy to verify that*

$$\varphi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right].$$

Therefore

$$(\varphi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}\left(e^{iu} - 1\right)\right].$$

and X *is infinitely divisible with* $X \stackrel{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ *and*

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}\right).$$

Example 2.14 *(Compound Poisson random variable) Let* $\{Z(n), n \in \mathbb{N}\}$ *be a sequence of i.i.d. random variables with law* μ_Z. *Let* $N \sim Po(\lambda)$ *and independent of the* $Z(n)'s$. *Define*

$$X = Z(1) + Z(2) + \cdots + Z(N).$$

Let us prove that, for each $u \in \mathbb{R}^d$,

$$\varphi_X(u) = \exp\left[\int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy)\right]. \tag{2.1}$$

Indeed, by conditioning, we have

$$\begin{aligned}\varphi_X(u) &= E\left[e^{i(u,X)}\right] = \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(N))} | N = n\right] P[N = n] \\ &= \sum_{n=0}^{\infty} E\left[e^{i(u,Z(1)+Z(2)+\cdots+Z(n))}\right] \frac{\lambda^n}{n!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda \varphi_Z(u))^n}{n!} = \exp\left[\lambda\left(\varphi_Z(u) - 1\right)\right].\end{aligned}$$

Therefore, with $\varphi_Z(u) = \int_{\mathbb{R}^d} e^{i(u,y)} \mu_Z(dy)$, *we obtain (2.1). We denote the Compound Poisson by* $X \sim Po(\lambda, \mu_Z)$. *We have*

$$(\varphi_X(u))^{\frac{1}{n}} = \exp\left[\frac{\lambda}{n}\left(\varphi_Z(u) - 1\right)\right]$$

and X *is infinitely divisible with* $X \overset{d}{=} Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)}$ *and*

$$Y_j^{(n)} \sim Po\left(\frac{\lambda}{n}, \mu_Z\right).$$

Exercise 2.15 *Show that if* $X \sim N(m, A)$, *where* A *is a* $d \times d$ *strictly positive-definite symmetric covariance matrix:* $A = E\left[(X - m)(X - m)^T\right]$ *then* $\varphi_X(u) = \exp\left(i(m, u) - \frac{1}{2}(u, Au)\right)$

Exercise 2.16 *Let* $d = 1$. *Show that if* $X \sim Po(\lambda)$ *then* $\varphi_X(u) = \exp\left[\lambda\left(e^{iu} - 1\right)\right]$.

2.2 The Lévy-Khintchine formula

Definition 2.17 *Let* ν *be a Borel measure defined on* $\mathbb{R}^d - \{0\}$. *We say that* ν *is a Lévy measure if*

$$\int_{\mathbb{R}^d - \{0\}} \left(|y|^2 \wedge 1\right) \nu(dy) < \infty \tag{2.2}$$

Note that $\varepsilon^2 \leq |y|^2 \wedge 1$ when $0 < \varepsilon \leq 1$ and $|y| \geq \varepsilon$. Therefore, by (2.2), we have that

$$\nu\left[(-\varepsilon, \varepsilon)^c\right] < \infty, \qquad \text{for all } \varepsilon > 0.$$

The condition (2.2) is equivalent to

$$\int_{\mathbb{R}^d - \{0\}} \frac{|y|^2}{1 + |y|^2} \nu(dy) < \infty.$$

One can assume that $\nu(\{0\}) = 0$ and then ν is defined on $\mathbb{R}^d$.

Theorem 2.18 *(Lévy-Khintchine): A probability measure μ on $\mathbb{R}^d$ is infinitely divisible if exists a vector $b \in \mathbb{R}^d$, a $d \times d$ positive definite symmetric matrix A and a Lévy measure ν on the set $\mathbb{R}^d - \{0\}$ such that, for all $u \in \mathbb{R}^d$:*

$$\varphi_\mu(u) = \exp\left\{ i(b, u) - \frac{1}{2}(u, Au) + \right. \\ \left. + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1 - i(u, y)\chi_{\widehat{B}}(y) \right] \nu(dy) \right\}, \qquad (2.3)$$

where $\widehat{B} = B_1(0) = \left\{ y \in \mathbb{R}^d : |y| < 1 \right\}$.

On the other hand, any function of the form (2.3) is a characteristic function of an infinitely divisible measure on $\mathbb{R}^d$.

(b, A, ν) are the characteristics of the infinitely divisible random variable X, $\eta := \log(\varphi_\mu)$ is the Lévy symbol or characteristic exponent or Lévy exponent

$$\eta(u) = i(b, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1 - i(u, y)\chi_{\widehat{B}}(y) \right] \nu(dy).$$

We will not prove the first part of the theorem, but we prove the second part.
Proof. (Part 2) Our aim is to prove that the r.h.s of (2.3) is a characteristic function.

i) Let $\{U(n), n \in \mathbb{N}\} \subset \mathbb{R}^d$ be a sequence of measurable (Borel) sets such that $U(n) \searrow 0$ and define

$$\varphi_n(u) = \exp\left\{ i\left(b - \int_{U(n)^c \cap \widehat{B}} y\nu(dy), u \right) - \frac{1}{2}(u, Au) + \right. \\ \left. + \int_{U(n)^c} \left(e^{i(u,y)} - 1 \right) \nu(dy) \right\}.$$

ii) Clearly, φ_n is the convolution of a Gaussian distribution with a compound Poisson distribution. Therefore, by Proposition 2.5, it is infinitely divisible.

iii) Clearly,

$$\varphi_\mu(u) = \lim_{n\to\infty} \varphi_n(u).$$

iv) In order to prove that φ_μ is a characteristic function, we will aplly Lévy's continuity theorem (see below) and therefore we only need to prove that $\psi_\mu(u)$ is continuous at zero, with

$$\begin{aligned}\psi_\mu(u) &= \int_{\mathbb{R}^d-\{0\}} \left[e^{i(u,y)} - 1 - i(u,y)\chi_{\widehat{B}}(y)\right]\nu(dy) \\ &= \int_{\widehat{B}} \left(e^{i(u,y)} - 1 - i(u,y)\right)\nu(dy) + \\ &+ \int_{\widehat{B}^c} \left(e^{i(u,y)} - 1\right)\nu(dy).\end{aligned}$$

v) By Taylor's theorem, the Cauchy-Schwarz inequality and dominated convergence, we have:

$$\begin{aligned}|\psi_\mu(u)| &\le \frac{1}{2}\int_{\widehat{B}} |(u,y)|^2\,\nu(dy) + \int_{\widehat{B}^c} \left|e^{i(u,y)} - 1\right|\nu(dy) \\ &\le \frac{|u|^2}{2}\int_{\widehat{B}} |y|^2\,\nu(dy) + \int_{\widehat{B}^c} \left|e^{i(u,y)} - 1\right|\nu(dy) \to 0 \text{ as } u \to 0.\end{aligned}$$

vi) It is now easy to verify directly that μ is infinitely divisible. ■

Some remarks:

- The techique of taking the limits of sequences composed of sums of gaussians with independent compound Poissons is very important.
- The cut-off function $c(y) = y\chi_{\widehat{B}}$ in (2.3) could be replaced by other $c(y)$ such that $e^{i(u,y)} - 1 - i(u, c(y))$ is an ν-integrable function for each $u \in \mathbb{R}^d$. For instance, we could have $c(y) = \frac{y}{1+|y|^2}$.
- The Gaussian case corresponds to $b = m$ (mean), A =covariance matrix, $\nu = 0$.
- The Poisson case corresponds to $b = 0$, $A = 0$, $\nu = \lambda\delta_1$.
- The compound Poisson case corresponds to $b = 0$, $A = 0$, $\nu = \lambda\mu$, where $\lambda > 0$ and μ is a probability measure on $\mathbb{R}^d$.

- All infinitely divisible distributions can be constructed as weak limits of convolutions of Gaussians with independent Poisson processes. In fact, they can be obtained as weak limits of Compound Poissons only.

Theorem 2.19 *Every infinitely divisible probability measure is the weak limit of an appropriate sequence of compound Poisson distributions.*

Proof. If μ is an infinitely divisible distribution then $\varphi^{\frac{1}{n}}$ is the characteristic function of $\mu^{\frac{1}{n}}$ and

$$\varphi_n(u) = \exp\left\{n\left[\varphi^{\frac{1}{n}}(u) - 1\right]\right\}$$

is the characteristic function of a compound Poisson distribution. Moreover,

$$\begin{aligned}\varphi_n(u) &= \exp\left\{n\left[e^{\frac{1}{n}\log(\varphi(u))} - 1\right]\right\} = \\ &= \exp\left\{\log(\varphi(u)) + n\ o\left(\frac{1}{n}\right)\right\} \to \varphi(u).\end{aligned}$$

Therefore, by the Glivenko Theorem (see [1]), μ is the weak limit of the compound Poisson distributions. ■

The Glivenko Theorem used in the previous proof says that: if φ_n and φ are the characteristic functions of μ_n and μ then

$$\varphi_n(u) \to \varphi(u) \text{ for all } u \in \mathbb{R}^d \Longrightarrow \mu_n \xrightarrow{w} \mu \text{ (weak convergence).}$$

Corollary 2.20 *The set of all infinitely divisible probability measures on $\mathbb{R}^d$ coincides with the weak closure of the set of all compound Poisson distributions on $\mathbb{R}^d$.*

Proof. Use the theorem and the property: If $\{\mu_n, n \in \mathbb{N}\}$ are infinitely divisible and $\mu_n \xrightarrow{w} \mu$ then μ is infinitely divisible. ■

2.3 Stable random variables

The class of stable distributions is an important subclass of infinitely divisible distributions. Let $d = 1$ and $\{Y_n, n \in \mathbb{N}\}$ be a sequence of i.i.d. random variables. Let us state the central limit problem. Define the partial sums of random variables:

$$S_n = \frac{Y_1 + \cdots + Y_n - b_n}{\sigma_n},$$

where $\{b_n, n \in \mathbb{N}\}$ is a real number sequence and $\{\sigma_n, n \in \mathbb{N}\}$ is a sequence of positive numbers.

- Problem: When exists a random variable X such that

$$\lim_{n\to\infty} P(S_n \leq x) = \lim_{n\to\infty} P(X \leq x) \qquad ? \tag{2.4}$$

In that case, S_n converges in distribution to X?

The classical central limit theorem gives a positive answer to these question with $b_n = nm$ and $\sigma_n = \sqrt{n}\sigma$. Then $X \sim N(m, \sigma^2)$.

A random variable is said to be stable if it can be obtained as a limit of the type (2.4). This is equivalent to the following definition.

Definition 2.21 *A random variable X is said to be stable if exist sequences $\{c_n, n \in \mathbb{N}\}, \{d_n, n \in \mathbb{N}\}$ with each $c_n > 0$, such that*

$$X_1 + \cdots + X_n \stackrel{d}{=} c_n X + d_n, \tag{2.5}$$

where $X_1, \ldots, X_n$ are independent and have the distribution of X. In particular, is said to be strictly stable if each $d_n = 0$.

In fact, one can prove that if X is stable then $\sigma_n = \sigma n^{\frac{1}{\alpha}}$ with $0 < \alpha \leq 2$. The parameter α is called the index of stability. The eq. (2.5) is equivalent to

$$\varphi_X(u)^n = e^{iud_n} \varphi_X(c_n u).$$

All stable random variables are infinitely divisible (it is a trivial consequence of (2.5)).

Theorem 2.22 *If X is a stable random variable then:*

1. *when $\alpha = 2$, $X \sim N(b, A)$*
2. *when $\alpha \neq 2$, $A = 0$ and*

$$\nu(dx) = \begin{cases} \frac{c_1}{x^{1+\alpha}} dx & \text{if } x > 0 \\ \frac{c_2}{|x|^{1+\alpha}} dx & \text{if } x < 0. \end{cases}, \quad \text{where } c_1, c_2 \geq 0 \text{ and } c_1 + c_2 > 0.$$

For a proof of this theorem, see [6].

Theorem 2.23 *A random variable X is stable if and only if exist $\sigma > 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that*

1. *when $\alpha = 2$,*

$$\varphi_X(u) = \exp\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right);$$

2. *when* $\alpha \neq 1, 2$

$$\varphi_X(u) = \exp\left(i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right)$$

3. *when* $\alpha = 1$,

$$\varphi_X(u) = \exp\left(i\mu u - \sigma |u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]\right)$$

For a proof, see [6], p. 86.
Some remarks about α-stable distributions:

- $E[X^2] < \infty$ if and only if $\alpha = 2$ (X is Gaussian).
- $E[|X|] < \infty$ if and only if $1 < \alpha \leq 2$.
- All stable random variables X have densities f_X. In general, can be expressed in series form, but in 3 cases, we have a closed form.
- **Normal distribution:** $\alpha = 2$ and $X \sim N(\mu, \sigma^2)$.
- **Cauchy distribution:** $\alpha = 1$, $\beta = 0$, $f_X(x) = \frac{\sigma}{\pi\left[(x-\mu)^2+\sigma^2\right]}$.
- **Lévy distribution**: $\alpha = \frac{1}{2}$, $\beta = 1$,

$$f_X(x) = \left(\frac{\sigma}{2\pi}\right)^{\frac{1}{2}} \frac{1}{(x-\mu)^{\frac{3}{2}}} \exp\left[\frac{\sigma}{-2(x-\mu)}\right] \quad \text{for } x > \mu.$$

Exercise 2.24 *Let* X *and* Y *be independent standard Gaussian random variables (with 0 mean). Show that* Z *has a Cauchy distribution, where* $Z = X/Y$ *if* $Y \neq 0$ *and* $Z = 0$ *if* $Y = 0$.

Remark 2.25 *If* X *is stable and symmetric then*

$$\varphi_X(u) = \exp(-\rho^\alpha |u|^\alpha) \quad \textit{for all } 0 < \alpha \leq 2,$$

where $\rho = \sigma$ *for* $0 < \alpha < 2$ *and* $\rho = \frac{\sigma}{\sqrt{2}}$ *when* $\alpha = 2$.

An Important feature of stable laws is the following one: when $\alpha \neq 2$ the decay of the tails is polynomial. This slow decay implies the existence of "fat tails". If $\alpha = 2$ the decay is exponential and there are no "fat tails". The decay can be described by

$$P[X > x] \sim \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}x} \quad \text{if } \alpha = 2,$$
$$\lim_{x\to\infty} x^\alpha P[X > x] \sim C_\alpha \frac{1+\beta}{2}\sigma^\alpha \text{ if } \alpha \neq 2, \text{ with } C_\alpha > 1.$$

All the previous results can be generalized to random variables with values in $\mathbb{R}^d$. Just replace $X_1, \ldots, X_n, X$ and each d_n in (2.5) by vectors and adapt the previous theorems. Note that when $\alpha \neq 2$ and $d > 1$, then the Lévy measure is given by

$$\nu(dx) = \frac{c}{|x|^{d+\alpha}} dx, \quad \text{where } c > 0.$$

2.4 Lévy Processes

Definition 2.26 *Let $L = (L(t); t \geq 0)$ be a stochastic process. The process L has independent increments if for each $n \in \mathbb{N}$ and each sequence $0 \leq t_1 < t_2 < \ldots < t_{n+1} < \infty$, the random variables $(L(t_{j+1}) - L(t_j); 1 \leq j \leq n)$ are independent. The process L has stationary increments if $L(t_{j+1}) - L(t_j) \overset{d}{=} L(t_{j+1} - t_j) - L(0)$.*

Definition 2.27 *We say that L is a Lévy process if*
(1) $L(0) = 0$ (a.s),
(2) L has independent and stationary increments,
(3) L is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t\to s} P(|L(t) - L(s)| > a) = 0.$$

Conditions (1) and (2) imply that (3) is equivalent to $\lim_{t\searrow 0} P(|L(t)| > a) = 0$. The sample paths (trajectories) of L are the maps $t \to L(t)(\omega)$ for each $\omega \in \Omega$.

Proposition 2.28 *If L is a Lévy process, then $L(t)$ is infinitely divisible for each $t \geq 0$.*

Proof. For each $n \in \mathbb{N}$,

$$L(t) = Y_1^{(n)}(t) + \cdots + Y_n^{(n)}(t),$$

where

$$Y_j^{(n)}(t) = L\left(\frac{jt}{n}\right) - L\left(\frac{(j-1)\,t}{n}\right).$$

By condition (2), these $Y_j^{(n)}(t)'s$ are iid random variable and therefore, $L\,(t)$ is infinitely divisible. ■

Theorem 2.29 *If L is a Lévy process, then*

$$\varphi_{L(t)}\,(u) = e^{t\eta(u)},$$

for each $u \in \mathbb{R}^d$, where η is the characteristic exponent (or Lévy symbol) of $L\,(1)$.

Proof. Define $\varphi_u\,(t) = \varphi_{L(t)}\,(u)$. Then, by condition (2),

$$\begin{aligned}\varphi_u\,(t+s) &= E\left[e^{i(u,L(t+s)-L(s)+L(s))}\right] \\ &= E\left[e^{i(u,L(t+s)-L(s))}\right] E\left[e^{i(u,L(s))}\right] = \varphi_u\,(t)\,\varphi_u\,(s)\,.\end{aligned}$$

On the other hand, by condition (1), $\varphi_u\,(0) = 1$. The map $t \to \varphi_u\,(t)$ is clearly continuous. The unique continuous function that satisfies all these conditions is of the form $\varphi_u\,(t) = e^{t\alpha(u)}$. But $L(1)$ is also infinitely divisible and therefore $\varphi_u\,(t) = e^{t\eta(u)}$ and $\alpha\,(u) = \eta\,(u)$. ■

Exercise 2.30 *Prove that if L is stochastically continuous, then the map $t \to \varphi_{L(t)}\,(u)$ is also continuous for each u. (Hint: see [1], pages 43-44).*

The Lévy-Khintchine formula for a Lévy Process $L = (L(t); t \geq 0)$ is given by

$$\varphi_{L(t)}\,(u) = E\left[e^{i(u,L(t))}\right] = \exp\left\{t\left[i\,(b,u) - \frac{1}{2}\,(u,Au) + \right.\right.$$
$$\left.\left. + \int_{\mathbb{R}^d-\{0\}} \left[e^{i(u,y)} - 1 - i\,(u,y)\,\chi_{\widehat{B}}\,(y)\right]\nu\,(dy)\right]\right\}, \qquad (2.6)$$

for each $t \geq 0$ and $u \in \mathbb{R}^d$. The characteristics (b, A, ν) correspond to $L\,(1)$.

Exercise 2.31 *Let L and Y be stochastically continuous processes. Show that their sum $L+Y$ is also stochastically continuous (hint: use the elementary inequality: $P\,(|A+B| > C) \leq P\left(|A| > \frac{C}{2}\right) + P\left(|B| > \frac{C}{2}\right)$ with A, B random variables).*

2.5 Examples of Lévy processes

We now present several examples of Lévy processes.

Example 2.32 *A Brownian motion in $\mathbb{R}^d$ is clearly a Lévy process B for which*

1. $B(t) \sim N(0, tI)$.
2. *B has continuous trajectories.*

From property 1, we obtain

$$\varphi_{B(t)}(u) = \exp\left\{-\frac{1}{2}t|u|^2\right\}.$$

Some of the main properties of Brownian motion (with $d = 1$) are

- *Brownian motion is locally Hölder continuous with exponent α for every $0 < \alpha < \frac{1}{2}$:*

$$|B(t)(\omega) - B(t)(\omega)| \leq K(T, \omega)|t - s|^{\alpha},$$

for all $0 \leq s < t \leq T$.

- *The sample paths (trajectories) $t \to B(t)(\omega)$ are a.s. nowhere differentiable.*

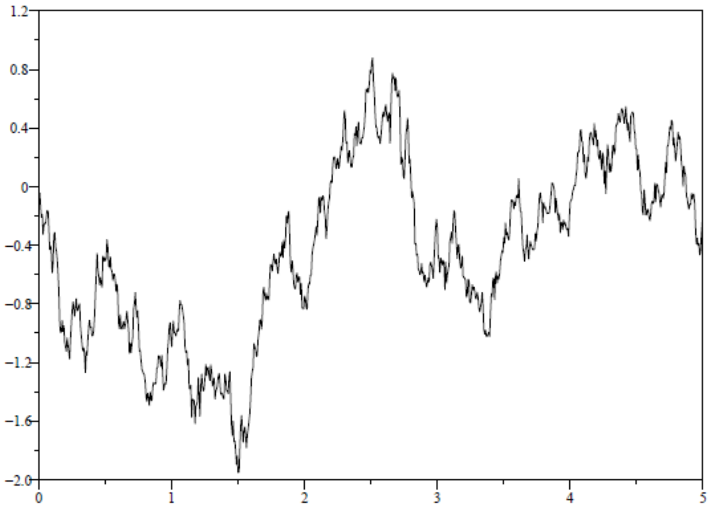
- *Consider a sequence $(t_n, n \in \mathbb{N})$ with $t_n \nearrow \infty$. Then, we have*

$$\liminf_{n\to\infty} B(t_n) = -\infty \quad a.s.$$
$$\limsup_{n\to\infty} B(t_n) = +\infty \quad a.s.$$

- *Law of iterated logarithm:*

$$\limsup_{t\searrow 0} \frac{B(t)}{\left(2t\log\left(\log\left(\frac{1}{t}\right)\right)\right)^{\frac{1}{2}}} = 1 \quad a.s.$$

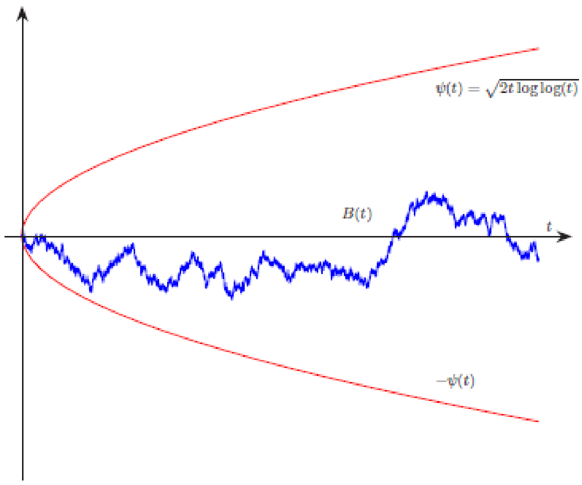
- *Simulated path of standard Brownian motion:*



A path of the Brownian motion (from [2])

- *Law of the iterated logarithm:*

$$\limsup_{t\to\infty}\frac{B(t)}{(2t\log(\log t))^{\frac{1}{2}}}=1,\ \liminf_{t\to\infty}\frac{B(t)}{(2t\log(\log t))^{\frac{1}{2}}}=-1\quad a.s.$$



The iterated logarithm law

Example 2.33 *(Brownian motion with drift) Given a non-negative definite symmetric $d\times d$ matrix A, let σ be such that $\sigma\sigma^T=A$. Let $b\in\mathbb{R}^d$ and denote by B be a Brownian motion in $\mathbb{R}^m$. The process C defined by*

$$C(t)=bt+\sigma B(t) \tag{2.7}$$

is a Lévy process that satisfies $C(t) \sim N(tb, tA)$. *Moreover,* C *is a Gaussian process (with Gaussian finite dimensional distributions). The process* C *is known as a Brownian motion with drift. The characteristic exponent of* C *is*

$$\eta_C(u) = i(b,u) - \frac{1}{2}(u, Au).$$

A Lévy process has continuous trajectories if and only if it is a Brownian motion with drift.

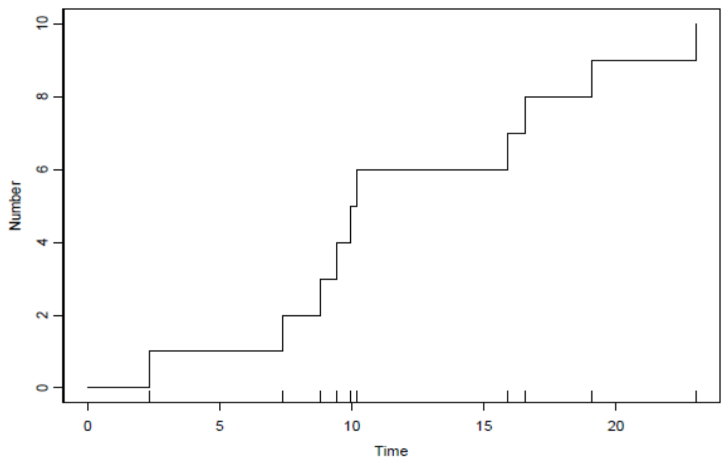
Example 2.34 *(Poisson process) Let* $N(t) \sim Po(\lambda t)$ *be a process taking values in* $\mathbb{N}_0$ *with*

$$P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Let us define the non-negative r.v. $\{T(n), n \in \mathbb{N}_0\}$ *(waiting times),* $T(0) = 0$,

$$T(n) = \inf\{t \geq 0 : N(t) = n\}.$$

The random variable $T(n)$ *has a gamma distribution. The inter-arrival times* $T(n) - T(n-1)$ *are iid random variables with exponential distribution of mean* $1/\lambda$.



A path of a Poisson process (from [2])

Example 2.35 *(Compensated Poisson process) Consider the process* $\widetilde{N} = \left(\widetilde{N}(t), t \geq 0\right)$, *where* $\widetilde{N}(t) = N(t) - \lambda t$. *Note that* $E\left[\widetilde{N}(t)\right] = 0$ *and* $E\left[\left(\widetilde{N}(t)\right)^2\right] = \lambda t$.

Example 2.36 *Consider the sequence of iid random variables* $\{Z(n), n \in \mathbb{N}\}$, *with values in* $\mathbb{R}^d$ *and with law* μ_Z. *Let* N *be a Poisson process with intensity parameter* λ *and independent of the* $Z(n)'$ *s. Then, we define the compound Poisson process by*

$$Y(t) = \sum_{n=1}^{N(t)} Z(n), \tag{2.8}$$

and $Y(t) \sim \pi(\lambda t, \mu_Z)$. *The characteristic exponent is*

$$\eta_Y(u) = \int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy).$$

The sample paths of Y *are constant (piecewice) with jumps occurring at times* $T(n)$, *but now the jump sizes are random.*

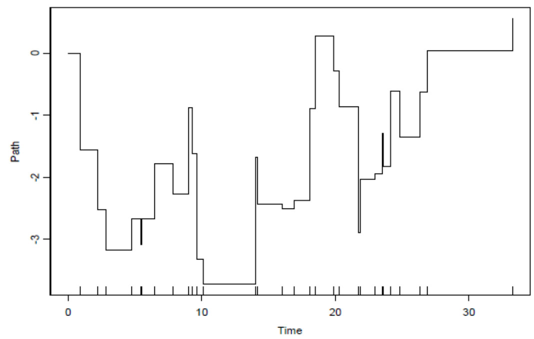


Figure 3. Simulation of a compound Poisson process with $N(0,1)$ summands($\lambda = 1$).

A trajectory of a compound Poisson process (from [2])

Let C be a Gaussian Lévy process and let Y be a compound Poisson process (which is independent of C). Define

$$L(t) = C(t) + Y(t).$$

Then L is clearly a Lévy process with Lévy characteristic exponent

$$\eta_L(u) = i(m,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy).$$

Let T_n represent the time of jump n. We have the interlacing process:

$$L(t) = \begin{cases} C(t) \quad \text{for } 0 \le t < T_1, \\ C(T_1) + Z_1 \quad \text{for } t = T_1, \\ L(T_1) + C(t) - C(T_1) \quad \text{for } T_1 \le t < T_2, \\ L(T_2-) + Z_2 \text{ for } t = T_2, \\ etc... \end{cases}$$

2.6 Stable Lévy processes

Definition 2.37 *A stable Lévy process is a Lévy process L, where each $L(t)$ is a stable random variable.*

Theorem 2.38 *If L is a α-stable Lévy process then the characteristic exponent of L is (with $\sigma > 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$) given by*

1. *when $\alpha = 2$,*
$$\eta_L(u) = i\mu u - \frac{1}{2}\sigma^2 u^2;$$
2. *when $\alpha \neq 1, 2$*
$$\eta_L(u) = i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right];$$
3. *when $\alpha = 1$,*
$$\eta_L(u) = i\mu u - \sigma |u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right].$$

An important case is the case of rotationally invariant stable Lévy processes, where
$$\eta_L(u) = -\sigma^\alpha |u|^\alpha, \qquad 0 < \alpha \leq 2.$$

These processes are important because they are self-similar. A process $Y = (Y(t), t \geq 0)$ is said to be self-similar, with positive Hurst index H, if $(Y(at), t \geq 0)$ and $\left(a^H Y(t), t \geq 0\right)$ have the same finite dimensional distributions for each $a \geq 0$. By examining the characteristic functions, one can prove that a rotationally invariant stable Lévy process is self-similar with Hurst parameter $H = 1/\alpha$. Moreover, one can also prove that a Lévy process L is self-similar if and only if each random variable $L(t)$ is strictly stable.

2.7 Subordinators

Definition 2.39 *A subordinator is simply a one-dimensional Lévy process wich is increasing with probability 1.*

A subordinator is an appropriate random model for stochastic time evolution. If $T = (T(t), t \geq 0)$ is a subordinator then $T(t) \geq 0$ a.s. and $T(t_1) \leq T(t_2)$ a.s. if $t_1 \leq t_2$.

Theorem 2.40 *Let T be a subordinator. Then its characteristic exponent has the form*

$$\eta_T(u) = i(b,u) + \int_{(0,\infty)} \left(e^{iuy} - 1\right) \lambda(dy), \tag{2.9}$$

where $b \geq 0$, and the Lévy measure λ satisfies: $\lambda(-\infty, 0) = 0$ and $\int_{(0,\infty)} (y \wedge 1) \lambda(dy) < \infty$.

Conversely, any mapping $\eta : \mathbb{R} \to \mathbb{C}$ of the form (2.9) is the characteristic exponent of a subordinator.

The characteristics of the subordinator T are (b, λ).

For each $t \geq 0$, the function $u \to E\left[e^{iuT(t)}\right]$ can be analytically continued to the set $\{iu, u > 0\}$, and we get the Laplace transform of the distribution:

$$E\left[e^{-uT(t)}\right] = e^{-t\psi(u)},$$

where

$$\psi(u) = -\eta(iu) = bu + \int_{(0,\infty)} \left(1 - e^{-yu}\right) \lambda(dy). \tag{2.10}$$

The function ψ is known as the Laplace exponent of the distribution.

We now present some important examples of subordinators.

Example 2.41 *A Poisson process is clearly a subordinator*

Example 2.42 *The compound Poisson process (2.8) is clearly a subordinator if and only if the random variables for the jump sizes are non-negative.*

Example 2.43 *One can prove (using the usual calculus) that (for $0 < \alpha < 1$ and $u \geq 0$)*

$$u^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-ux}\right) \frac{dx}{x^{1+\alpha}}.$$

Then, by (2.10) and the properties of a stable Lévy process, exists an α-stable subordinator with Laplace exponent $\psi(u) = u^\alpha$. The characteristics of T are clearly $(0, \lambda)$, where

$$\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}.$$

When we consider the analytic continuation of this, in order to obtain the Lévy characteristic exponent, we obtain $\mu = 0$, $\beta = 1$ and $\sigma^\alpha = \cos(\alpha\pi/2)$.

Example 2.44 *The $\left(\frac{1}{2}\right)$-stable subordinator has a probability density function associated to the Lévy distribution (with $\mu = 0$ and $\sigma = \frac{t^2}{2}$):*

$$f_{T(t)}(s) = \left(\frac{t}{2\sqrt{\pi}}\right) s^{-\frac{3}{2}} \exp\left(\frac{-t^2}{4s}\right).$$

With some calculus, one can show that

$$E\left[e^{-uT(t)}\right] = \int_0^\infty e^{-us} f_{T(t)}(s)\, ds = e^{-tu^{\frac{1}{2}}}.$$

Exercise 2.45 *Show that $E\left[e^{-uT(t)}\right] = \int_0^\infty e^{-us} f_{T(t)}(s)\, ds = e^{-tu^{\frac{1}{2}}}$. (Hint: Differentiate $g_t(u) = \int_0^\infty e^{-us} f_{T(t)}(s)\, ds$ with respect to u and use the substitution $x = \frac{t^2}{4us}$ in order to get the differential equation $g_t'(u) = -\frac{t}{2\sqrt{u}} g_t(u)$. Via the substitution $y = \frac{t}{2\sqrt{s}}$ we see that $g_t(0) = 1$ and one can prove the result).*

Example 2.46 *The $\left(\frac{1}{2}\right)$-stable subordinator can be represented by the hitting time of the Brownian motion*

$$T(t) = \inf\left\{s > 0 : B(s) = \frac{t}{\sqrt{2}}\right\}. \tag{2.11}$$

We can generalize the Lévy subordinator by replacing the Brownian motion in the hitting time by the Gaussian process $C(t) = B(t) + \mu t$, and the inverse Gaussian subordinator is defined by

$$T_\delta(t) = \inf\{s > 0 : C(s) = \delta t\}$$

where $\delta > 0$. Note that $t \to T_\delta(t)$ is the generalized inverse of a Gaussian process. This means that the Gaussian distribution describes a Brownian motion value at a fixed time, and the inverse Gaussian describes the distribution of the time that a Brownian Motion with drift takes to reach a fixed value δt .

Using martingale methods, one can show that for each $t, u > 0$,

$$E\left[e^{-uT_\delta(t)}\right] = \exp\left(-t\delta\sqrt{2u + \mu^2} - \mu\right)$$

and $T(t)$ has a density:

$$f_{T_\delta(t)}(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \mu} s^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(t^2\delta^2 s^{-1} + \mu^2 s\right)\right],$$

for $s, t \geq 0$. A random variable with probability density $f_{T_\delta(1)}$ is usually called an inverse Gaussian random variable. For this distribution we use the notation $IG(\delta, \mu)$.

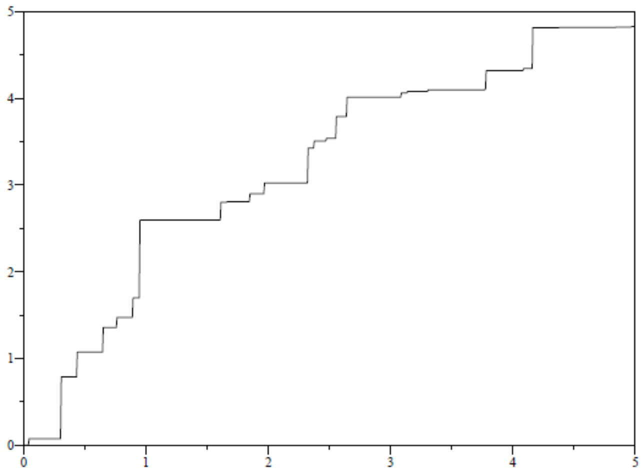


Figure 2.1: A path of a Gamma subordinator (from [2])

Example 2.47 *Let $T(t)$ be a Gamma process with parameters $a, b > 0$, such that $T(t)$ has a probability density function*

$$f_{T(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx}, \quad x \geq 0.$$

Using some calculus, we can show that

$$\int_0^\infty e^{-ux} f_{T(t)}(x)\, dx = \left(1 + \frac{u}{b}\right)^{-at} = \exp\left(-ta \log\left(1 + \frac{u}{b}\right)\right)$$
$$= \exp\left(-t \int_0^\infty \left(1 - e^{-ux}\right) ax^{-1} e^{-bx} dx\right).$$

Therefore, by (2.10), $T(t)$ is a subordinator with characteristics $b = 0$ and $\lambda(dx) = ax^{-1}e^{-bx}dx$. This subordinator is called the Gamma subordinator.

Subordinators are often used to model stochastic time changes. Let L be a Lévy process and T be an independent subordinator. Define a new process

$$Z(t) = L(T(t)).$$

Theorem 2.48 *Z is a Lévy process*

For a proof of thsi theorem, see [1], pags. 56-58.

Proposition 2.49

$$\eta_Z = -\psi_T \circ (-\eta_L).$$

Proof. Let $p_{T(t)}$ be the law associated to the subordinator $T(t)$. Then

$$\begin{aligned}
E\left[e^{t\eta_{Z(t)}(u)}\right] &= E\left(e^{i(u,Z(t))}\right) = E\left(e^{i(u,X(T(t)))}\right) \\
&= \int E\left(e^{i(u,X(T(s)))}\right) p_{T(t)}\left(ds\right) \\
&= \int e^{s\eta_X(u)} p_{T(t)}\left(ds\right) \\
&= E\left[e^{-(-\eta_X(u))T(t)}\right] \\
&= e^{-t\psi_T(-\eta_X(u))}. \qquad \blacksquare
\end{aligned}$$

■

Example 2.50 *(Brownian motion and (2α)-stable motion) Let T be an α-stable subordinator (with $0 < \alpha < 1$) and let L be a Brownian motion with covariance matrix $A = 2I$ (independent of T). Then*

$$\psi_T\left(s\right) = s^{\alpha}, \qquad \eta_L\left(u\right) = -\left|u\right|^2$$

and therefore, by Proposition 2.49,

$$\eta_Z\left(u\right) = -\left|u\right|^{2\alpha}$$

and Z is a 2α stable process. When $d = 1$ and T is a Lévy subordinator, the process Z is a Cauchy process and each $Z(t)$ has symmetric Cauchy distribution ($\mu = 0$ and $\sigma = 1$). Moreover, by (2.11), the Cauchy process can be constructed from two indepedent Brownian motions.

Example 2.51 *(the variance gamma process) Let $Z(t) = B(T(t))$, where T is a gamma subordinator and B is a Brownian motion. Then the Lévy process Z is called a variance-gamma process. It has this name, because in this process we replace the variance of the Brownian motion by a gamma random variable. Then, by Proposition 2.49, we have*

$$\Phi_{Z(t)}\left(u\right) = E\left[e^{uiZ(t)}\right] = \left(1 + \frac{u^2}{2b}\right)^{-at},$$

where a and b are the parameters of the gamma process.

Manipulating characteristic functions, it is possible to show that:

$$Z(t) = G(t) - L(t)$$

where G and L are gamma subordinators with parameters $\sqrt{2b}$ and a (this can be interpreted as the difference of independent "gains" and "losses"). From this representation, it is possible to show that $Z(t)$ has a Lévy density:

$$g_\nu(x) = \frac{a}{|x|^1}\left(e^{\sqrt{2b}x}\chi_{(-\infty,0)}(x) + e^{-\sqrt{2b}x}\chi_{(0,\infty)}(x)\right), \quad a > 0.$$

Exercise 2.52 *Prove that if T is a gamma subordinator and B is a Brownian motion then $Z(t) = B(T(t))$ is a Lévy process with characteristic function*

$$\Phi_{Z(t)}(u) = E\left[e^{uiZ(t)}\right] = \left(1 + \frac{u^2}{2b}\right)^{-at}.$$

Example 2.53 *The CGMY model (from the authors Carr, Geman, Madan and Yor) is a generalization of the variance gamma process, with Lévy density*

$$g_\nu(x) = \frac{a}{|x|^{1+\alpha}}\left(e^{b_1 x}\chi_{(-\infty,0)}(x) + e^{-b_2 x}\chi_{(0,\infty)}(x)\right), \quad a > 0,\ 0 \le \alpha < 2,\quad b_1, b_2 \ge 0.$$

When $b_1 = b_2 = 0$, this corresponds to stable Lévy processes. The exponential dampens the effects of large jumps.

Example 2.54 *Let $Z(t) = C(T(t)) + \mu t$, where $C(t) = B(t) + \beta t$ and the process T is an inverse Gaussian subordinator. Let α be such that $\alpha^2 \ge \beta^2$. Then Z has characteristic function ($\delta > 0$):*

$$\Phi_{Z(t)}(\alpha, \beta, \delta, \mu)(u) = \exp\left[\delta t\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right) + i\mu t u\right],$$

with probability density function

$$f_{Z(t)}(x) = C(\alpha, \beta, \delta, \mu; t)\, q\left(\frac{x - \mu t}{\delta t}\right)^{-1} K_1\left(\delta t \alpha q\left(\frac{x - \mu t}{\delta t}\right)\right) e^{\beta x},$$

where $q(x) = \sqrt{1 + x^2}$, $C(\alpha, \beta, \delta, \mu; t) = \pi^{-1}\alpha e^{\delta t\sqrt{\alpha^2 - \beta^2} - \beta\mu t}$ and K_1 is a Bessel function of the third kind.

Chapter 3

Stochastic calculus for Lévy processes

3.1 Martingales and random measures

Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space with filtration $(\mathcal{F}_t, t \geq 0)$.

Definition 3.1 *A stochastic process $X = (X(t), t \geq 0)$ is said to be adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ if each $X(t)$ is $\mathcal{F}_t$-measurable*

Any process X is clearly adapted to its natural filtration $\mathcal{F}_t^X := \sigma \{X(s), s \leq t\}$.

Definition 3.2 *An adapted process X is a Markov process if for all measurable bounded function f, we have (for $s \leq t$)*

$$E[f(X(t)) | \mathcal{F}_s] = E[f(X(t)) | X(s)] \qquad a.s.$$

For a Markov process, the past and the future of the process are independent if we know the present. The transition probabilities associated to a Markov process are

$$p_{s,t}(x, A) = P[X(t) \in A | X(s) = x].$$

Theorem 3.3 *If L is an adapted Lévy process, where each $L(t)$ has law q_t, then it is also a Markov process with associated transition probabilities*

$$p_{s,t}(x, A) = q_{t-s}(A - x).$$

Proof. By the stationarity of increments,

$$\begin{aligned} E\left[f\left(L\left(t\right)\right)|\mathcal{F}_s\right] &= E\left[f\left(L\left(s\right)+L\left(t\right)-L\left(s\right)\right)|\mathcal{F}_s\right] \\ &= \int_{\mathbb{R}^d} f\left(L\left(s\right)+y\right) q_{t-s}\left(dy\right). \end{aligned}$$

Hence,

$$E\left[f\left(L\left(t\right)\right)|\mathcal{F}_s\right] = E\left[f\left(L\left(t\right)\right)|L_s\right]$$

and the transition probabilities are obtained for $f = \chi_A$ and $p_{s,t}\left(x, A\right) = \int_{\mathbb{R}^d} \chi_A\left(x+y\right) q_{t-s}\left(dy\right) = q_{t-s}\left(A-x\right)$. ■

Definition 3.4 *The process X is a martingale if X is adapted to $\left(\mathcal{F}_t, t \geq 0\right)$, $E\left[|X\left(t\right)|\right] < \infty$ for all $t \geq 0$ and*

$$E\left[X\left(t\right)|\mathcal{F}_s\right] = X_s \quad a.s \quad \text{for all } s < t.$$

Theorem 3.5 *If a Lévy process L is adapted, has finite first moment and zero mean, then it is a martingale.*

Proof. Clearly, L is adapted, $E\left[|L\left(t\right)|\right] < \infty$ for all $t \geq 0$ and

$$\begin{aligned} E\left[L\left(t\right)|\mathcal{F}_s\right] &= E\left[L\left(s\right)+L\left(t\right)-L\left(s\right)|\mathcal{F}_s\right] \\ &= L\left(s\right)+E\left[L\left(t\right)-L\left(s\right)\right] = L\left(s\right). \end{aligned}$$

■

We now present several examples of Lévy processes that are also martingales:

1. $\sigma B\left(t\right)$, $B\left(t\right)$ d-dim. BM and σ an $r \times d$ matrix.
2. $\widetilde{N}\left(t\right)$ - compensated Poisson process
3. $\exp\left\{i\left(u, L\left(t\right)\right) - t\eta\left(u\right)\right\}$ where $u \in \mathbb{R}^d$ is fixed and L is a Lévy process with Lévy symbol η.
4. $|\sigma B\left(t\right)|^2 - trace\left(A\right)t$, with $A = \sigma^T\sigma$
5. $\left[\widetilde{N}\left(t\right)\right]^2 - \lambda t$

Exercise 3.6 *If L is a Lévy process, show that* $\exp\left\{i\left(u, L\left(t\right)\right) - t\eta\left(u\right)\right\}$ *is a martingale.*

Let us now introduce the definition of càdlàg function.

Definition 3.7 *A function $f : \mathbb{R}^+ \to \mathbb{R}$ is a càdlàg function ("continue à droite et limité à gauche") if it is right continuous with left limits.*

Consider the notation: $f(t-) := \lim_{s \uparrow t} f(s)$ and $\Delta f(t) := f(t) - f(t-)$.

If f is càdlàg then $\#\{0 \leq t \leq T : \Delta f(t) \neq 0\}$ is at most countable. If the filtration satisfies the "usual hypothesis", then every Lévy process can be replaced by a càdlàg modification, which is also a Lévy process (for a proof, see Theorem 2.1.8, pag 87 in [1]).

The usual hypothesis for a filtration $(\mathcal{F}_t, t \geq 0)$ are:

1. (completeness): $\mathcal{F}_0$ contains all sets of P-measure 0.

2. (right continuity): $\mathcal{F}_t = \mathcal{F}_{t+}$ where $\mathcal{F}_{t+} = \cap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$.

We will assume that:

- $(\Omega, \mathcal{F}, P)$ will be a fixed filtered probability space with a filtration $(\mathcal{F}_t, t \geq 0)$ which satisfies the "usual hypotheses".

- Every Lévy process L will be a $\mathcal{F}_t$-adapted process with càdlàg trajectories.

- $L(t) - L(s)$ is independent of the σ–algebra $\mathcal{F}_s$, for each $s < t$.

Let us also recall that given two processes, $(X(t), t \geq 0)$ and $(Y(t), t \geq 0)$, we say that Y is a modification of X if, for each $t \geq 0$, $P[X(t) \neq Y(t)] = 0$. As a consequence, X and Y share the same finite dimensional distributions.

The jump process ΔX associated to X is defined by

$$\Delta X(t) = X(t) - X(t-).$$

Theorem 3.8 *If N is an increasing Lévy process with values on the natural numbers, such that $\Delta N(t) \in \{0, 1\}$, then N is a Poisson process.*

For a proof of this theorem see [1].

Lemma 3.9 *If L is a Lévy process, then with fixed $t > 0$, we have $\Delta L(t) = 0$ (a.s.).*

Proof. Let $(t(n); n \in N)$ be a sequence of positive numbers with $t(n) \uparrow t$ as $n \to \infty$. The process L has càdlàg paths. Hence, $\lim_{n \to \infty} L(t(n)) = L(t-)$. By the stochastic continuity condition (in the Lévy process definition), $L(t(n))$ converges (in the probability sense) to $L(t)$. Hence, it has a subsequence converging to $L(t)$ a.s. Then, by the uniqueness of the limits $L(t) = L(t-)$ (a.s.) and $\Delta L(t) = 0$ (a.s.). ■

In general, the analytic difficulty in manipulating Lèvy processes has to do with the fact that is possible to have

$$\sum_{0\le s\le t} |\Delta L(s)| = \infty \quad \text{a.s.}$$

To overcome this difficulty, we will use the fact that

$$\sum_{0\le s\le t} |\Delta L(s)|^2 < \infty \quad \text{a.s.}$$

In order to count jumps of a specified size, define for a borelian set $A \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$, the counting measure

$$\begin{aligned} N(t,A) &= \#\left\{0 \le s \le t : \Delta L\left(s\right) \in A\right\} \\ &= \sum_{0\le s\le t} \chi_A\left(\Delta L(s)\right) \end{aligned}$$

For each $\omega \in \Omega$, $t \ge 0$, the map $A \to N(t,A)$ is a counting measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$. We denote by $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ the $\sigma-$algebra of Borel measurable sets in $\mathbb{R}^d - \{0\}$. Then

$$E\left[N(t,A)\right] = \int N(t,A)\left(\omega\right) dP\left(\omega\right)$$

is a measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$. Let us consider $\mu\left(\cdot\right) = E\left[N(1,\cdot)\right]$, which is a measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ called the intensity measure. This measure gives the mean number of jumps until time 1.

We say that $A \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ is bounded from below if $0 \notin \overline{A}$, where $\overline{A}$ is the closure of set A.

Lemma 3.10 *If A is bounded from below then $N(t,A) < \infty$ a.s. for all $t \ge 0$.*

Sketch of the Proof: Define the stopping times $(T_n^A, n \in \mathbb{N})$ by

$$T_1^A = \inf\left\{t > 0 : \Delta L\left(t\right) \in A\right\}$$

and

$$T_n^A = \inf\left\{t > T_{n-1}^A : \Delta L\left(t\right) \in A\right\}.$$

Since L has càdlàg paths, we have that $T_1^A > 0$ a.s. and $\underset{n\to\infty}{lim}\ T_n^A = \infty$ a.s. Otherwise, one could prove that the set of all jumps with size in A would have an accumulation point and this is impossible for càdlàg processes (see the proof of Theorem 2.8.1 in appendix 2.8 of [1]). Moreover,

$$N(t, A) = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{T_n^A \leq t\}} < \infty \quad \text{a.s.}$$

Note that if A is not bounded below, then the Lemma no longer holds. Indeed, there is an accumulation of many small jumps.

Theorem 3.11 *1. If A is bounded from below, then $(N(t, A), t \geq 0)$ is a Poisson process with intensity parameter $\mu(A)$.*

2. If $A_1, \ldots A_m \in \mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$ are disjoint sets then the random variables $N(t, A_1), \ldots, N(t, A_m)$ are independent.

For a proof of this theorem, see pages 101-103 of [1].

A consequence of the theorem is that $\mu(A) < \infty$ if A is bounded from below.

- Main properties of N:

  1. For each t and $\omega \in \Omega$, $N(t, \cdot)(\omega)$ is a counting measure on $\mathcal{B}\left(\mathbb{R}^d - \{0\}\right)$.
  2. For each A bounded from below, $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A) = E\left[N(1, A)\right]$.
  3. The compensated $(\widetilde{N}(t, A), t \geq 0)$ is a martingale-valued measure, considering $\widetilde{N}(t, A) = N(t, A) - t\mu(A)$, for A bounded from below, i.e. for fixed A bounded below, $(\widetilde{N}(t, A), t \geq 0)$ is a martingale.

3.2 Poisson integrals

Let f be a measurable function and A be bounded from below. Then we may define the Poisson integral of the function f as the random finite sum

$$\int_A f(x) N(t, dx)(\omega) = \sum_{x \in A} f(x) N(t, \{x\})(\omega),$$

where $\{x\}$ are the jump sizes of the process (in A), i.e.

$$N(t, \{x\}) \neq 0 \Longleftrightarrow \Delta X(u) = x$$

for some $0 \leq u \leq t$. We have that

$$\int_A f(x) N(t, dx)$$

is a random variable and defines a stochastic process as t changes. We have also that

$$\int_A f(x) N(t, dx) = \sum_{0 \leq u \leq t} f(\Delta L(u)) \mathbf{1}_A (\Delta L(u)).$$

Theorem 3.12 *Let A be bounded from below. Then:*

1. $\left(\int_A f(x) N(t, dx), t \geq 0\right)$ is a compound Poisson process with characteristic function

$$\exp\left(t \int_{\mathbb{R}^d} \left(e^{i(u,x)} - 1\right) \mu_{f,A}(dx)\right),$$

where $\mu_{f,A}(B) = \mu\left(A \cap f^{-1}(B)\right)$ for $B \in \mathcal{B}\left(\mathbb{R}^d\right)$.

2. If $f \in L^1(A, \mu_A)$ then (μ_A is the measure μ restricted to A):

$$\mathbb{E}\left[\int_A f(x) N(t, dx)\right] = t \int_A f(x) \mu(dx).$$

3. If $f \in L^2(A, \mu_A)$ then

$$\operatorname{Var}\left(\left|\int_A f(x) N(t, dx)\right|\right) = t \int_A |f(x)|^2 \mu(dx).$$

Sketch of the proof: 1. Assume that f is the simple function

$$f = \sum_{j=1}^{n} c_j \mathbf{1}_{A_j}$$

(with the A_j's disjoint). Then, by part 2 of the previous theorem, we have that

$$E\left[\exp\left\{i\left(u, \int_A f(x) N(t, dx)\right)\right\}\right] = \prod_{j=1}^{n} E\left[\exp\left\{i\left(u, \int_A c_j N(t, A_j)\right)\right\}\right]$$

$$= \prod_{j=1}^{n} \exp\left\{t\left(e^{i(u,c_j)} - 1\right) \mu(A_j)\right\} = \exp\left\{t\left(e^{i(u,f(x))} - 1\right) \mu(dx)\right\}.$$

For a function $f \in L^1(A, \mu_A)$, there is a sequence of simple functions that converges to f in L^1. Therefore, a subsequence of this sequence converges to f a.s. If we pass to the limit in this subsequence, we obtain the result. Parts

2. and 3. follow from 1. by differentiation (moments can be obtained from the characteristic function by the formula $E\left[X^k\right] = (-i)^k \varphi^{(k)}(0)$)■

From the part (2) of the previous theorem, it follows that a Poisson integral will not have a finite mean if $f \notin L^1(A, \mu)$. For $f \in L^1(A, \mu_A)$, we define the compensated Poisson integral by

$$\int_A f(x)\, \widetilde{N}(t, dx) = \int_A f(x)\, N(t, dx) - t \int_A f(x)\, \mu(dx).$$

The process $\left(\int_A f(x)\, \widetilde{N}(t, dx), t \geq 0\right)$ is a martingale. By the previous theorem, we have that

$$E\left[\exp\left\{i\left(u, \int_A f(x)\, \widetilde{N}(t, dx)\right)\right\}\right]$$
$$= \exp\left(t \int_{\mathbb{R}^d} \left(e^{i(u,x)} - 1 - i(u,x)\right) \mu_{f,A}(dx)\right)$$

and if $f \in L^2(A, \mu_A)$, then

$$E\left[\left|\int_A f(x)\, \widetilde{N}(t, dx)\right|^2\right] = t \int_A |f(x)|^2 \mu(dx).$$

3.3 The Lévy-Itô decomposition

Let $\mathcal{P} = \{a = t_1 < t_2 < \cdots < t_n < t_{n+1} = b\}$ be a partition of $[a, b] \subset \mathbb{R}$, with diameter $\delta = \max_{1 \leq i \leq n} |t_{i+1} - t_i|$.

Definition 3.13 *The variation $Var_{\mathcal{P}}[g]$ of a function g over the partition $\mathcal{P}$ is defined by*

$$Var_{\mathcal{P}}[g] := \sum_{i=1}^{n} |g(t_{i+1}) - g(t_i)|.$$

Definition 3.14 *If $V[g] := \sup_{\mathcal{P}} Var_{\mathcal{P}}[g] < \infty$, we say g has finite variation on $[a, b]$.*

If g is defined on $\mathbb{R}$ (or $\mathbb{R}^+$), we say it has finite variation if it has finite variation for each compact interval. Every non-decreasing function g has finite variation.

Functions of finite variation are very inimportant for integration: if we propose g as an integrator, in order to define the Riemann-Stieltjes integral $\int_I f dg$ for a continuous function f, a necessary and sufficient condition for obtaining $\int_I f dg$ as the limit of Riemann sums is the finite variation of g.

Definition 3.15 *A stochastic process $(X(t), t \geq 0)$ is of finite variation if the trajectories $(X(t)(\omega), t \geq 0)$ have finite variation for a.a. $\omega \in \Omega$.*

Example 3.16 *Let N be a Poisson random measure, with intensity μ. Consider a measurable function f and a set A bounded from below. Then, the process*

$$Y(t) = \int_A f(x) N(t, dx).$$

has finite variation on $[0, t]$ for each $t \geq 0$. Indeed:

$$Var_{\mathcal{P}}[Y] \leq \sum_{0 \leq s \leq t} |f(\Delta X(s))| \mathbf{1}_A(\Delta X(s)) < \infty \qquad a.s. \ ,$$

where $X(t) = \int_A xN(t, dx)$ for each $t \geq 0$.

The following are necessary and sufficient conditions for a Lévy process to have finite variation:

1. there is no Brownian component ($A = 0$ in the formula of Lévy-Khinchine), and
2. $$\int_{|x|<1} |x| \nu(dx) < \infty.$$

For a set A bounded from below,

$$\int_A xN(t, dx) = \sum_{0 \leq s \leq t} \Delta L(s) \mathbf{1}_A(\Delta L(s)).$$

gives the sum of all the jumps with size in A, until time t. If L has càdlàg paths then this sum is a finite random sum. In particular,

$$\int_{|x| \geq 1} xN(t, dx)$$

is finite (the sum of big jumps sizes). It is a compound Poisson process, has finite variation and the moments can be finite or infinite. On the other hand,

$$L(t) - \int_{|x| \geq 1} xN(t, dx)$$

is a Lévy process, which has finite moments of all orders.

If L is a Lévy process with bounded jumps then we have $E(|L(t)|^m) < \infty$ for all $m \in \mathbb{N}$ (for a proof, see pages 118-119 of [1]).

For small jumps, let us consider compensated Poisson integrals (which are martingales) for A bounded below:

$$M(t, A) := \int_A x\widetilde{N}(t, dx).$$

Consider the "ring-sets"

$$B_m := \left\{ x \in \mathbb{R}^d : \frac{1}{m+1} < |x| \le \frac{1}{m} \right\},$$
$$A_n := \bigcup_{m=1}^{n} B_m.$$

We can define

$$\int_{|x|<1} x\widetilde{N}(t, dx) := (L^2 \text{ limit}) \lim_{n\to\infty} M(t, A_n).$$

Therefore, $\int_{|x|<1} x\widetilde{N}(t, dx)$ is a martingale because is the L^2 limit of a sequence of martingales. Taking the limit in

$$E\left[\exp\left\{ i\left(u, \int_{A_n} x\widetilde{N}(t, dx)\right)\right\}\right] = \exp\left(t\int_{\mathbb{R}^d} \left(e^{i(u,x)} - 1 - i(u, x)\right)\mu_{x,A_n}(dx)\right),$$

we obtain

$$E\left[\exp\left\{ i\left(u, \int_{|x|<1} x\widetilde{N}(t, dx)\right)\right\}\right]$$
$$= \exp\left(t\int_{|x|<1} \left(e^{i(u,x)} - 1 - i(u, x)\right)\mu(dx)\right)$$

Consider now the process

$$B_A(t) = L(t) - bt - \int_{|x|<1} x\widetilde{N}(t, dx) - \int_{|x|\ge 1} xN(t, dx),$$

where $b = \mathbb{E}\left(L(1) - \int_{|x|\ge 1} xN(1, dx)\right)$. Then B_A is a centered martingale with continuous paths and has covariance matrix A. Therefore, by the Lévy characterization of Brownian motion, the process B_A is a Brownian motion with covariance (matrix) A.

We have proved the famous Lévy-Itô decomposition for Lévy processes.

Theorem 3.17 *(Lévy-Itô decomposition) If L is a Lévy process, then exist a vector $b \in \mathbb{R}^d$, a Brownian motion with covariance matrix A, denoted by B_A, and an independent Poisson random measure N, defined on $\mathbb{R}^+ \times \left(\mathbb{R}^d - \{0\}\right)$, such that*

$$L(t) = bt + B_A(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx). \qquad (3.1)$$

and all the processes in (3.1) are independent.

The Lévy-Khintchine formula can be deduced as a corollary of the Lévy-Itô decomposition.

Corollary 3.18 *(Lévy-Khintchine formula) If L is a Lévy process then*

$$E\left[e^{i(u,L(t))}\right] = \exp\left\{t\left[i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d-\{0\}} \left[e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{|x|<1}(x)\right]\nu(dx)\right]\right\}$$

- The intensity measure μ is equal to the Lévy measure ν for L.
- $\int_{|x|<1} x\widetilde{N}(t, dx)$ gives the compensated sum of small jumps sizes, which is an L^2-martingale.
- $\int_{|x|\geq 1} xN(t, dx)$ is the sum of large jumps (compound Poisson process - it may fail to have finite moments).

A Lévy process has finite variation if its Lévy-Itô decomposition is

$$\begin{aligned} L(t) &= \gamma t + \int_{x\neq 0} xN(t, dx) \\ &= \gamma t + \sum_{0\leq s\leq t} \Delta L(s), \end{aligned}$$

where $\gamma = b - \int_{|x|<1} x\nu(dx)$.

The terms of the Lévy-Itô decomposition have a financial interpretation:

- if the intensity measure (μ or ν) is infinite, the stock price has "infinite activity" and this means that flutuations and small jumpy movements are caused by the interaction of supply and demand shocks.

- if the intensity measure (μ or ν) is finite, we have "finite activity" and big movements or jumps occur in the market, which can be caused by sudden events like a major natural disaster.
- If a pure jump Lévy process (no Brownian component) has finite activity then it has finite variation. The converse statement is false.
- In the rhs of (3.1), the first three components have finite moments of all orders. Indeed, if a Lévy process does not have a finite moment, this is caused by the "large jumps"/"finite activity" part $\int_{|x|\geq 1} xN(t,dx)$.

One can proce that $E\left[|L(t)|^n\right] < \infty$ if and only if $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$.

3.4 Stochastic integration

Definition 3.19 *A stochastic process $Y = \{Y(t), t \geq 0\}$ is a semimartingale if it is an adapted process that can be decomposed as*

$$Y = Y(0) + M(t) + F(t), \tag{3.2}$$

where M is a local martingale and F is a finite variation and adapted process.

Semimartingales are "good integrators", in the sense that is the largest class of processes with respect to which the Itô integral can be properly defined. A Lévy process is a semimartingale. Indeed, by (3.1), it is clear that

$$M(t) = B_A(t) + \int_{|x|<1} x\widetilde{N}(t,dx),$$
$$F(t) = bt + \int_{|x|\geq 1} xN(t,dx).$$

are a martingale and a finite variation process.

Let $Y = M + F$ be a semimartingale. The stochastic integral with respect to Y can be represented in the form:

$$\int_0^t u(s)\,dY_s = \int_0^t u(s)\,dM_s + \int_0^t u(s)\,dF_s, \tag{3.3}$$

where $\int_0^t u(s)\,dF_s$ is defined by the usual Lebesgue-Stieltjes integral. In general, $\int_0^t u(s)\,dM_s$ requires a stochastic definition because, in general, M has infinite variation. We define, for $E \subset \mathbb{R}^d$,

$$\int_0^t \int_E u(s,x)\,M(ds,dx) = \int_0^t b(s)\,dB_s + \int_0^t \int_{E-\{0\}} u(s,x)\,\widetilde{N}(ds,dx), \tag{3.4}$$

where $b(s) := u(s, 0)$.

Let $\mathcal{P}$ be the smallest σ-algebra such that all the mappings $u : [0, T] \times E \times \Omega \to \mathbb{R}$ satisfying (1) and (2) below are measurable:

1. For each t, $(x, \omega) \to u(t, x, \omega)$ is $\mathcal{B}(E) \times \mathcal{F}_t$ measurable.

2. For each x and ω, $t \to u(t, x, \omega)$ is left continuous.

Definition 3.20 *$\mathcal{P}$ is called the predictable σ-algebra. A $\mathcal{P}$-measurable mapping (or process) is said to be predictable (predictable process).*

Let $\mathcal{H}_2$ be the linear space of mappings (or processes) $u : [0, T] \times E \times \Omega \to \mathbb{R}$ which are predictable and satisfy

$$\int_0^T \int_{E-\{0\}} \mathbb{E}\left[|u(t, x)|^2\right] \nu(dx)\, dt < \infty, \tag{3.5}$$

$$\int_0^T \mathbb{E}\left[|u(t, 0)|^2\right] dt < \infty. \tag{3.6}$$

A simple process u is a process of the form

$$u = \sum_{j=1}^{m} \sum_{k=1}^{n} u_k(t_j)\, \mathbf{1}_{(t_j, t_{j+1}]} \mathbf{1}_{A_k}. \tag{3.7}$$

We denote by $\mathcal{S}$ the class of simple processes. A simple process is predictable and its stochastic integral is defined by

$$I(u) = \sum_{j=1}^{m} \sum_{k=1}^{n} u_k(t_j)\, M((t_j, t_{j+1}], A_k), \tag{3.8}$$

where $M((t_j, t_{j+1}], A_k) = M(t_{j+1}, A_k) - M(t_j, A_k) = [B(t_{j+1}) - B(t_j)]\, \delta_0(A_k) + \left[\widetilde{N}(t_{j+1}, A_k - 0) - \widetilde{N}(t_j, A_k - 0)\right]$, where $\delta_0(\cdot)$ is a Dirac measure.

Lemma 3.21 *If u is simple then*

$$\mathbb{E}[I(u)] = 0,$$

$$\mathbb{E}\left[(I(u))^2\right] = \int_0^T \int_{E-\{0\}} \mathbb{E}\left[|u(t, x)|^2\right] \nu(dx)\, dt + \delta_0(E) \int_0^T \mathbb{E}\left[|u(t, 0)|^2\right] dt \tag{3.9}$$

Exercise 3.22 *For a simple process u, show that $\mathbb{E}[I(u)] = 0$.*

I is a linear isometry from the space $\mathcal{S}$ (set of simple processes) into $L^2(\Omega)$ and since $\mathcal{S}$ is dense in $\mathcal{H}_2$, I can be extended to $\mathcal{H}_2$ and it is a isometry of $\mathcal{H}_2$ into $L^2(\Omega)$. For $u \in \mathcal{H}_2$ we define the stochastic integral by

$$I_t(u) = \int_0^t \int_E u(s,x)\, M(ds,dx)$$

and

$$\int_0^t \int_E u(s,x)\, M(ds,dx) = \lim_{n\to\infty} (L^2) \int_0^t \int_E u_n(s,x)\, M(ds,dx), \qquad (3.10)$$

where $\{u_n, n \in \mathbb{N}\}$ is a sequence (of simple processes) such that $u_n \to u$ in $\mathcal{H}_2$.

The stochastic integral $I_t(u)$, with $u \in \mathcal{H}_2$, satisfies the properties:

1. I_t is a linear operator.
2. $\mathbb{E}[I(u)] = 0, \mathbb{E}\left[(I(u))^2\right] = \int_0^T \int_{E-\{0\}} \mathbb{E}\left[|u(t,x)|^2\right] \nu(dx)\, dt + \delta_0(E) \int_0^T \mathbb{E}\left[|u(t,0)|^2\right] dt.$
3. $\{I_t(u),\ t \in [0,T]\}$ is $\{\mathcal{F}_t\}$ adapted.
4. $\{I_t(u),\ t \in [0,T]\}$ is a square-integrable martingale.

Sketch of the Proof of (3): Let $(u_n, n \in \mathbb{N})$ be a sequence of simple processes in $\mathcal{H}_2$ that converges to u. Then $(I_t(u_n), t \geq 0)$ is adapted and $I_t(u_n) \longrightarrow I_t(u)$ in L^2. Therefore, there is a subsequence $(u_{n_k}; n_k \in \mathbb{N})$ such that $I_t(u_{n_k}) \longrightarrow I_t(u)$ a.s. as $n_k \to \infty$. Therefore $\{I_t(u),\ t \geq 0\}$ is $\{\mathcal{F}_t\}$ adapted. ■

Sketch of the Proof of (4): Let u be a simple process in $\mathcal{H}_2$ and choose $0 < s = t_l < t_{l+1} < t$. Then $I_t(u) = I_s(u) + I_{s,t}(u)$ and by property (3), we have that

$$\mathbb{E}_s(I_t(u)) = I_s(u) + \mathbb{E}_s(I_{s,t}(u))$$

Moreover,

$$\begin{aligned}\mathbb{E}_s(I_{s,t}(u)) &= \mathbb{E}_s\left(\sum_{j=l+1}^{m} \sum_{k=1}^{n} u_k(t_j)\, M((t_j, t_{j+1}], A_k)\right)\\ &= \sum_{j=l+1}^{m} \sum_{k=1}^{n} \mathbb{E}_s(u_k(t_j))\, \mathbb{E}_s\left[M((t_j, t_{j+1}], A_k)\right] = 0.\end{aligned}$$

Therefore, $\mathbb{E}_s(I_t(u)) = I_s(u)$ and $\{I_t(u),\ t \geq 0\}$ is a martingale.

Now, let $(u_n, n \in \mathbb{N})$ be a sequence of simple processes converging to u in L^2. It can be proved that (see [1]) $\mathbb{E}_s(I_t(u_n)) \to \mathbb{E}_s(I_t(u))$ in L^2 and therefore $\mathbb{E}_s(I_t(u)) = I_s(u)$ is a square-integrable martingale■

3.5 Lévy-Type stochastic integrals

The integral of a predictable process $w(t,x)$ with respect to the compound Poisson process $P_t = \int_A xN(t,dx)$ is defined by (with A bounded below)

$$\int_0^T \int_A w(t,x)\, N(dt,dx) = \sum_{0\le s\le T} w(s,\Delta P_s)\, \mathbf{1}_A(\Delta P_s). \tag{3.11}$$

We can also define

$$\int_0^T \int_A h(t,x)\, \widetilde{N}(dt,dx) = \int_0^T \int_A h(t,x)\, N(dt,dx) - \int_0^T \int_A h(t,x)\, \nu(dx)\, dt \tag{3.12}$$

if h is predictable and satisfies (3.5).

We say Y is a Lévy type stochastic integral if

$$\begin{aligned} Y_t = Y_0 &+ \int_0^t b(s)\, ds + \int_0^t u(s)\, dB_s + \int_0^t \int_{|x|<1} h(s,x)\, \widetilde{N}(ds,dx) \\ &+ \int_0^t \int_{|x|\ge 1} w(s,x)\, N(ds,dx), \end{aligned} \tag{3.13}$$

where we assume that the processes b, u, h and w are predictable and satisfy the appropriate integrability conditions.

Then Y is a semimartingale and equation (3.13) can be written as

$$dY_t = b(t)\, dt + u(t)\, dB_t + \int_{|x|<1} h(t,x)\, \widetilde{N}(dt,dx) + \int_{|x|\ge 1} w(t,x)\, N(dt,dx).$$

Let L be a Lévy process with Lévy triplet (b,c,ν) and let X be a predictable left-continuous process satisfying (3.5). Then, we can construct a Lévy stochastic integral Y_t by

$$dY_t = X_t dL_t.$$

The stochastic integrals can be defined in an extended space $\mathcal{P}_2(T,E)$, where $\mathcal{H}_2 \subset \mathcal{P}_2(T,E)$. The space $\mathcal{P}_2(T,E)$ is defined as the set of mappings $u : [0,T]\times E\times\Omega \to \mathbb{R}$, such that:

1) u is predictable

2)

$$P\left[\int_0^T \int_{E-\{0\}} |u(t,x)|^2\, \nu(dx)\, dt < \infty\right] = 1, \tag{3.14}$$

$$P\left[\int_0^T |u(t,0)|^2\, dt < \infty\right] = 1. \tag{3.15}$$

If $u \in \mathcal{P}_2(T, E)$ then $\{I_t(u), \ t \geq 0\}$ is not necessarily a martingale (it can be just a local martingale). If $E = \{0\}$, we use the notation $\mathcal{P}_2(T)$ for $\mathcal{P}_2(T, E)$. Therefore, $\mathcal{P}_2(T)$ is the set of all predictable mappings $b : [0, T] \times \Omega \to \mathbb{R}$ such that

$$P\left[\int_0^T |b(t)|^2 \, dt < \infty\right] = 1$$

We say Y is a Lévy type stochastic integral if

$$\begin{aligned} Y_t^i = Y_0 &+ \int_0^t b^i(s)\, ds + \int_0^t u_j^i(s)\, dB_s^j + \int_0^t \int_{|x|<1} h^i(s, x)\, \widetilde{N}(ds, dx) \\ &+ \int_0^t \int_{|x|\geq 1} w^i(s, x)\, N(ds, dx), \qquad i = 1, ..., d,\ j = 1, ..., m \end{aligned} \tag{3.16}$$

where $|b^i|^{\frac{1}{2}}$, $u_j^i \in \mathcal{P}_2(T)$ and $h^i \in \mathcal{P}_2(T, E)$ and w is predictable. With the stochastic differentials notation, in the one-dimensional case, we can write

$$\begin{aligned} dY(t) = b(t)\, dt + u(t)\, dB(t) &+ \int_{|x|<1} h(t, x)\, \widetilde{N}(dt, dx) \\ &+ \int_{|x|\geq 1} w(t, x)\, N(dt, dx). \end{aligned}$$

Let M be an adapted and left-continuous process. Then, we can define a new process $\{Z_t, t \geq 0\}$ by

$$dZ(t) = M(t)\, dY(t)$$

or

$$\begin{aligned} dZ(t) = {} & M(t)\, b(t)\, dt + M(t)\, u(t)\, dB(t) + M(t)\, h(t, x)\, \widetilde{N}(dt, dx) \\ & + M(t)\, w(t, x)\, N(dt, dx). \end{aligned}$$

Example 3.23 *Let X be a Lévy process with characteristics (b, A, ν) and Lévy-Itô decomposition*

$$X(t) = bt + B_A(t) + \int_{|x|<1} x\widetilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

Let $L \in \mathcal{P}_2(t)$ for all $t \geq 0$ and choose in (3.16) $u_j^i = A_j^i L$, $h^i = w^i = x^i L$. The process Y such that

$$dY(t) = L(t)\, dX(t)$$

is called a Lévy stochastic integral.

Example 3.24 *(Ornstein-Uhlenbeck (OU) process) Let X be a Lévy process. Consider the process*

$$Y(t) = e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)} dX(s),$$

where y_0 is fixed. This process can be used for volatility modelling in finance and is known as the Ornstein-Uhlenbeck (OU) process. In differential form, the OU process is the solution of the SDE

$$dY(t) = -\lambda Y(t)\, dt + dX(t),$$

which is known as the Langevin equation (is a stochastic differential equation). The Langevin equation is also a model for the physical Brownian motion. In this equation there is a viscous drag term and a stochastic term.

Exercise 3.25 *Prove that if X is a one-dimensional Brownian motion then the OU process $Y(t)$ is a Gaussian process with mean $e^{-\lambda t} y_0$ and variance $\frac{1}{2\lambda}\left(1 - e^{-2\lambda t}\right)$.*

3.6 Itô formula

Consider the Poisson stochastic integral

$$W(t) = W(0) + \int_0^t \int_A w(s,x)\, N(ds, dx),$$

with A bounded below and w predictable.

Lemma 3.26 *(Itô formula 1) If $f \in C(\mathbb{R})$ then*

$$f(W(t)) - f(W(0)) = \int_0^t \int_A \left[f(W(s-) + w(s,x)) - f(W(s-))\right] N(ds, dx) \quad a.s.$$

Proof. Let $Y(t) = \int_A x N(t, dx)$. The jump times of Y can be defined by $T_0^A = 0$, $T_n^A = \inf\left\{t > T_{n-1}^A; \Delta Y(t) \in A\right\}$. ■

Then

$$\begin{aligned}
f(W(t)) - f(W(0)) &= \sum_{0 \le s \le t} \left[f(W(s)) - f(W(s-))\right] \\
&= \sum_{n=1}^{\infty} \left[f\left(W\left(t \wedge T_n^A\right)\right) - f\left(W\left(t \wedge T_{n-1}^A\right)\right)\right] \\
&= \sum_{n=1}^{\infty} f\left(W\left(t \wedge T_n^A -\right) + w\left(t \wedge T_n^A, \Delta Y\left(t \wedge T_n^A\right)\right)\right) - f\left(W\left(t \wedge T_n^A -\right)\right) \\
&= \int_0^t \int_A \left[f(W(s-) + w(s,x)) - f(W(s-))\right] N(ds, dx).
\end{aligned}$$

Let M be a Itô process of the form

$$M^i(t) = \int_0^t u^i_j(s)\, dB^j(s) + \int_0^t b^i(s)\, ds,$$

with $u^i_j, |b^i|^{\frac{1}{2}} \in \mathcal{P}_2(t)$. Let us define the quadratic variation process:

$$\left[M^i, M^j\right](t) = \sum_{k=1}^{m} \int_0^t u^i_k(s)\, u^j_k(s)\, ds.$$

We now present the Itô formula for Brownian motion

Theorem 3.27 *(Itô formula 2) If $f \in C^2(\mathbb{R}^d)$ then*

$$f(M(t)) - f(M(0)) = \int_0^t \partial_i f(M(s))\, dM^i(s) + \frac{1}{2}\int_0^t \partial_i \partial_j f(M(s))\, d\left[M^i, M^j\right](s). \quad a.s.$$

For a proof of this theorem, see [1].

Consider now the Lévy-type stochastic integral

$$dY(t) = b(t)\, dt + u(t)\, dB(t) + \int_{|x|<1} h(t,x)\, \widetilde{N}(dt, dx) + \int_{|x|\geq 1} w(t,x)\, N(dt, dx),$$

where

$$dY_c(t) := b(t)\, dt + u(t)\, dB(t),$$

and

$$dY_d(t) := \int_{|x|<1} h(t,x)\, \widetilde{N}(dt, dx) + \int_{|x|\geq 1} w(t,x)\, N(dt, dx).$$

The Itô formula for this Lévy-type stochastic integral is given in the following theorem.

Theorem 3.28 *(Itô formula 3) If $f \in C^2(\mathbb{R}^d)$ then*

$$\begin{aligned} f(Y(t)) - f(Y(0)) &= \int_0^t \partial_i f(Y(s-))\, dY^i_c(s) + \frac{1}{2}\int_0^t \partial_i \partial_j f(Y(s-))\, d\left[Y^i_c, Y^j_c\right](s) \\ &+ \int_0^t \int_{|x|\geq 1} \left[f(Y(s-) + w(s,x)) - f(Y(s-))\right] N(ds, dx) \\ &+ \int_0^t \int_{|x|<1} \left[f(Y(s-) + h(s,x)) - f(Y(s-))\right] \widetilde{N}(ds, dx) \\ &+ \int_0^t \int_{|x|<1} \left[f(Y(s-) + h(s,x)) - f(Y(s-)) \right. \\ &\left. - h^i(s,x)\, \partial_i f(Y(s-))\right] \nu(dx)\, ds \end{aligned}$$

For a proof of this theorem, see [1]. This Itô formula can be presented in an alternative form.

Theorem 3.29 *(Itô formula 4): If $f \in C^2\left(\mathbb{R}^d\right)$ then*

$$f(Y(t)) - f(Y(0)) = \int_0^t \partial_i f(Y(s-))\, dY^i(s) + \frac{1}{2}\int_0^t \partial_i\partial_j f(Y(s-))\, d\left[Y_c^i, Y_c^j\right](s)$$
$$+ \sum_{0\le s\le t}\left[f(Y(s)) - f(Y(s-)) - \Delta Y^i(s)\,\partial_i f(Y(s-))\right].$$

The quadratic variation process for the process Y is given by

$$\left[Y^i, Y^j\right](t) = \left[Y_c^i, Y_c^j\right](t) + \sum_{0\le s\le t}\Delta Y^i(s)\,\Delta Y^j(s).$$

or by

$$\left[Y^i, Y^j\right](t) = \sum_{k=1}^m \int_0^t u_k^i(s)\, u_k^j(s)\, ds + \int_0^t\int_{|x|<1} h^i(s,x)\, h^j(s,x)\, \widetilde{N}(ds,dx)$$
$$+ \int_0^t\int_{|x|\ge 1} w^i(s,x)\, w^j(s,x)\, N(ds,dx). \tag{3.17}$$

Theorem 3.30 *If Y^1 and Y^2 are real valued Lévy-type stochastic integrals, then*

$$Y^1(t)\,Y^2(t) = Y^1(0)\,Y^2(0) + \int_0^t Y^1(s-)\, dY^2(s)$$
$$+ \int_0^t Y^2(s-)\, dY^1(s) + \left[Y^1, Y^2\right](t).$$

Proof. Take $f(x_1, x_2) = x_1 x_2$ and apply Itô's formula 4 in order to obtain

$$Y^1(t)\,Y^2(t) - Y^1(0)\,Y^2(0) = \int_0^t Y^1(s-)\, dY^2(s)$$
$$+ \int_0^t Y^2(s-)\, dY^1(s) + \left[Y_c^1, Y_c^2\right](t)$$
$$+ \sum_{0\le s\le t}\left[Y^1(s)\,Y^2(s) - Y^1(s-)\,Y^2(s-) - \Delta Y^1(s)\,Y^2(s-) - \Delta Y^2(s)\,Y^1(s-)\right]$$

and the result follows. ■

In differential form, this formula is

$$d\left(Y^1(t)\,Y^2(t)\right) = Y^1(t-)\,dY^2(t) + Y^2(t-)\,dY^1(t) + d\left[Y^1, Y^2\right](t).$$

The Itô correction term can be interpreted as the result of the product of stochastic differentials, as given in the following formulas (see (3.17)):

$$dB^i(t)\,dB^j(t) = \delta^{ij}dt,$$

$$N(dt, dx)\,N(dt, dy) = N(dt, dx)\,\delta(x - y),$$

the other differentials products vanish.

Chapter 4

Stochastic exponentials

Let $d = 1$ and consider the process $Z = (Z(t), t \geq 0)$, which is a solution of the SDE

$$dZ(t) = Z(t-)\, dY(t), \tag{4.1}$$

where Y is a Lévy-type stochastic integral, of the form

$$dY(t) = b(t)\, dt + u(t)\, dB(t) + \int_{|x|<1} h(t,x)\, \widetilde{N}(dt,dx) + \int_{|x|\geq 1} w(t,x)\, N(dt,dx).$$

The solution of (4.1) is the "stochastic exponential" or the "Doléans-Dade exponential"

$$Z(t) = \mathcal{E}_Y(t) = \exp\left\{ Y(t) - \frac{1}{2}[Y_c, Y_c](t) \right\} \prod_{0\leq s\leq t} (1 + \Delta Y(s))\, e^{-\Delta Y(s)}. \tag{4.2}$$

For financial applications, we will require that

$$\inf\{\Delta Y(t), t \geq 0\} > -1 \text{ a.s.} \tag{4.3}$$

Proposition 4.1 *If Y is a Lévy-type stochastic integral and (4.3) is satisfied, then each random variable $\mathcal{E}_Y(t)$ is a.s. finite.*

Exercise 4.2 *Prove the previous proposition (see Applebaum)*

Note that (4.3) also implies that $\mathcal{E}_Y(t) > 0$ a.s. The stochastic exponential $\mathcal{E}_Y(t)$ is the unique solution of SDE (4.1) which satisfies the initial condition $Z(0) = 1$ a.s. If (4.3) does not hold, then $\mathcal{E}_Y(t)$ may take negative values.

An alternative form for (4.2) is

$$\mathcal{E}_Y(t) = e^{S_Y(t)}, \tag{4.4}$$

where

$$
\begin{aligned}
dS_Y(t) &= u(t)\,dB(t) + \left(b(t) - \frac{1}{2}u(t)^2\right)dt \\
&+ \int_{|x|\geq 1} \log(1 + w(t,x))\,N(dt,dx) + \int_{|x|<1} \log(1 + h(t,x))\,\widetilde{N}(dt,dx) \\
&+ \int_{|x|<1} (\log(1 + h(t,x)) - h(t,x))\,\nu(dx)\,dt \qquad (4.5)
\end{aligned}
$$

Theorem 4.3

$$d\mathcal{E}_Y(t) = \mathcal{E}_Y(t)\,dY(t)$$

Exercise 4.4 *Prove the previous theorem by applying the Itô formula to (4.5).*

Example 4.5 *If $Y(t) = \sigma B(t)$, where $\sigma > 0$ and B is a Brownian motion, then*

$$\mathcal{E}_Y(t) = \exp\left\{\sigma B(t) - \frac{1}{2}\sigma^2 t\right\}.$$

Example 4.6 *If $Y = (Y(t), t \geq 0)$ is a compound Poisson process with $Y(t) = X_1 + \cdots + X_{N(t)}$, then*

$$\mathcal{E}_Y(t) = \prod_{i=1}^{N(t)} (1 + X_i)$$

Example 4.7 *If $Y(t) = \mu t + \sigma B(t) + J(t)$, where $\sigma > 0$, B is a Brownian motion, and $J = (J(t), t \geq 0)$ is a compound Poisson process $J(t) = X_1 + \cdots + X_{N(t)}$, then*

$$\mathcal{E}_Y(t) = \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right\}\prod_{i=1}^{N(t)}(1 + X_i).$$

Let X be a Lévy process with characteristics (b, σ, ν) and Lévy-Itô decomposition

$$X(t) = bt + \sigma B(t) + \int_{|x|<1} x\widetilde{N}(t,dx) + \int_{|x|\geq 1} xN(t,dx).$$

When can $\mathcal{E}_X(t)$ be written as $\exp(X_1(t))$ for a certain Lévy process X_1 and vice-versa? By (4.4) and (4.5) we have $\mathcal{E}_X(t) = e^{S_X(t)}$ with

$$S_X(t) = \sigma B(t) + \int_{|x|\geq 1} \log(1+x) N(t,dx) + \int_{|x|<1} \log(1+x) \widetilde{N}(t,dx) + t\left[b - \frac{1}{2}\sigma^2 + \int_{|x|<1} (\log(1+x) - x)\nu(dx)\right]. \tag{4.6}$$

Comparing the Lévy-Itô decomposition with (4.6), we have the following theorem.

Theorem 4.8 *If X is a Lévy process with characteristics (b,σ,ν), then $\mathcal{E}_X(t) = \exp(X_1(t))$ where X_1 is a Lévy process with characteristics (b_1,σ_1,ν_1) given by*

$$\nu_1 = \nu \circ f^{-1}, \quad f(x) = \log(1+x).$$
$$b_1 = b - \frac{1}{2}\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[\log(1+x)\mathbf{1}_{]-1,1[}(\log(1+x)) - x\mathbf{1}_{]-1,1[}(x)\right]\nu(dx),$$
$$\sigma_1 = \sigma.$$

Conversely, there exists a Lévy process X_2 with characteristics (b_2,σ_2,ν_2) such that we have $\exp(X(t)) = \mathcal{E}_{X_2}(t)$, where

$$\nu_2 = \nu \circ g^{-1}, \quad g(x) = e^x - 1$$
$$b_2 = b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}-\{0\}} \left[(e^x - 1)\mathbf{1}_{]-1,1[}(e^x - 1) - x\mathbf{1}_{]-1,1[}(x)\right]\nu(dx),$$
$$\sigma_2 = \sigma.$$

4.1 Exponential martingales

Consider the Lévy-type stochastic integral

$$dY(t) = b(t)\,dt + u(t)\,dB(t) + \int_{|x|<1} h(t,x)\widetilde{N}(dt,dx) + \int_{|x|\geq 1} w(t,x) N(dt,dx).$$

When is Y a martingale? Consider the following assumptions

- (M1) $\mathbb{E}\left[\int_0^t \int_{|x|\geq 1} |w(s,x)|^2 \nu(dx)\,ds\right] < \infty$ for each $t > 0$.

- (M2) $\int_0^t \mathbb{E}\left[|b(s)|\right] ds < \infty$ for each $t > 0$.

As a consequence of (M1) and the Cauchy-Schwarz inequality, we have

$$\int_0^t \int_{|x|\geq 1} |w(s,x)|\, \nu(dx)\, ds < \infty \quad a.s.$$

and

$$\int_0^t \int_{|x|\geq 1} w(s,x)\, N(ds,dx) = \int_0^t \int_{|x|\geq 1} w(s,x)\, \widetilde{N}(ds,dx) + \int_0^t \int_{|x|\geq 1} w(s,x)\, \nu(dx)\, ds.$$

The compensated integral is a martingale. Therefore, we have the following theorem.

Theorem 4.9 *Consider the assumptions (M1) and (M2). The process Y is a martingale if and only if*

$$b(t) + \int_{|x|\geq 1} w(t,x)\, \nu(dx) = 0 \quad \text{(a.s.) for a.a. } t \geq 0.$$

Let us consider the process $e^Y = \left(e^{Y(t)}, t \geq 0\right)$ .By Itô's formula, we have that

$$\begin{aligned} e^{Y(t)} &= 1 + \int_0^t e^{Y(s-)} u(s)\, dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} \left(e^{h(s,x)} - 1\right) \widetilde{N}(ds,dx) \\ &+ \int_0^t \int_{|x|\geq 1} e^{Y(s-)} \left(e^{w(s,x)} - 1\right) \widetilde{N}(ds,dx) \\ &+ \int_0^t e^{Y(s-)} \Bigg(b(s) + \frac{1}{2} u(s)^2 + \int_{|x|<1} \left(e^{h(s,x)} - 1 - h(s,x)\right) \nu(dx) \\ &+ \int_{|x|\geq 1} \left(e^{w(s,x)} - 1\right) \nu(dx) \Bigg) ds \end{aligned} \tag{4.7}$$

and therefore, we have the following theorem.

Theorem 4.10 *The process e^Y is a martingale if and only if*

$$\begin{aligned} & b(s) + \frac{1}{2} u(s)^2 + \int_{|x|<1} \left(e^{h(s,x)} - 1 - h(s,x)\right) \nu(dx) \\ &+ \int_{|x|\geq 1} \left(e^{w(s,x)} - 1\right) \nu(dx) = 0 \quad a.s. \text{ and for a.a. } s \geq 0. \end{aligned} \tag{4.8}$$

If e^Y is a martingale, then

$$e^{Y(t)} = 1 + \int_0^t e^{Y(s-)} u(s)\, dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} \left(e^{h(s,x)} - 1\right) \widetilde{N}(ds, dx) \tag{4.9}$$

$$+ \int_0^t \int_{|x|\geq 1} e^{Y(s-)} \left(e^{w(s,x)} - 1\right) \widetilde{N}(ds, dx).$$

If e^Y is a martingale then $\mathbb{E}\left[e^{Y(t)}\right] = 1$ for all $t \geq 0$ and the process e^Y is usually called an exponential martingale.

Example 4.11 *If Y is an Itô process of the form*

$$Y(t) = \int_0^t b(s)\, ds + \int_0^t u(s)\, dB(s),$$

then (4.8) is $b(t) = -\frac{1}{2} u(t)^2$ and

$$e^{Y(t)} = \exp\left(\int_0^t u(s)\, dB(s) - \frac{1}{2}\int_0^t u(s)^2\, ds\right).$$

4.2 Change of Measure and Girsanov Theorem

Let P and Q be two different probability measures. Denote by Q_t and P_t the measures restricted to $(\Omega, \mathcal{F}_t)$. Let e^Y be an exponential martingale and define Q_t by

$$\frac{dQ_t}{dP_t} = e^{Y(t)}.$$

Fix an interval $[0, T]$ and define $P = P_T$ and $Q = Q_T$.

Lemma 4.12 *$M = (M(t), 0 \leq t \leq T)$ is a Q-martingale if and only if $Me^Y = (M(t)e^{Y(t)}, 0 \leq t \leq T)$ is a P-martingale.*

Let Y be an Itô process (or Brownian integral) and

$$e^{Y(t)} = \exp\left(\int_0^t u(s)\, dB(s) - \frac{1}{2}\int_0^t u(s)^2\, ds\right).$$

Define a new process

$$B_Q(t) = B(t) - \int_0^t u(s)\, ds.$$

Theorem 4.13 *(Girsanov): B_Q is a Brownian motion under the probability measure Q.*

For a proof of the Girsanov theorem, see [1]. We now present a generalization of the Girsanov theorem.

Theorem 4.14 *(Girsanov II) Let M be a martingale given by*

$$M(t) = \int_0^t \int_A L(x,s)\, \widetilde{N}(ds,dx),$$

with L predictable and $L \in \mathcal{P}_2$. Then

$$N(t) = M(t) - \int_0^t \int_A L(s,x)\left(e^{h(s,x)} - 1\right)\nu(dx)\, ds$$

is a Q-martingale.

Chapter 5

Lévy Processes in Option Pricing

5.1 Option pricing

Consider that we have a risky asset or stock with price process $S = (S(t), t \geq 0)$. A contingent claim with maturity date T is a non-negative $\mathcal{F}_T$-measurable random variable Z, representing the payoff of a financial derivative. Z. For instance, for an European call option, we have

$$Z = \max\{S(T) - K, 0\}.$$

For an American call option, we have

$$Z = \sup_{0 \leq \tau \leq T} [\max\{S(\tau) - K, 0\}].$$

For an Asian option, we have

$$Z = \max\left\{\frac{1}{T}\int_0^T (S(t) - K)\, dt, 0\right\}.$$

In our market model, we assume that the interest rate r is constant. Define the discounted stock price process $\widetilde{S} = \left(\widetilde{S}(t), t \geq 0\right)$ by

$$\widetilde{S}(t) = e^{-rt} S(t).$$

Consider a portfolio $(\alpha(t), \beta(t))$, where $\alpha(t)$ is the number of shares and $\beta(t)$ is the number of riskless assets (bonds). The portfolio value at time t is given by the process

$$V(t) = \alpha(t) S(t) + \beta(t) A(t).$$

A portfolio is said to be a replicating portfolio if $V(T) = Z$ and is said to be a self-financing portfolio if

$$dV(t) = \alpha(t)\, dS(t) + r\beta(t)\, A(t)\, dt.$$

A market is said to be complete if every contingent claim can be replicated by a self-financing portfolio.

An arbitrage opportunity exists if the market allows some type of risk-free profit. More precisely, the market is arbitrage free if there exists no self-financing strategy for which $V(0) = 0$, $V(T) \geq 0$ and $P(V(T) > 0) > 0$.

Theorem 5.1 *(Fundamental Theorem of Asset Pricing I - discrete time) If the market is free of arbitrage opportunities, then there exists a probability measure Q, equivalent to P, such that the discounted price process $\widetilde{S}$ is a Q-martingale.*

One can prove a similar result in the continuous time case. However, one need more technical assumptions - like the stronger NFLVR assumption ("no free lunch with vanishing risk").

Theorem 5.2 *(Fundamental Theorem of Asset Pricing II) An arbitrage-free market is complete if and only if there exists a unique probability measure Q, equivalent to P, such that the discounted price process $\widetilde{S}$ is a Q-martingale.*

Such a measure Q is usually called an equivalent martingale measure or a risk-neutral measure. If Q exists and is not unique, the market is incomplete. If the market is complete, we have that

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q\left[Z|\mathcal{F}_t\right]$$

which is the arbitrage-free price, at time t, of the contingent claim or payoff Z.

5.2 Stock price as a Lévy process

Consider the return of the risky asset in a time interval of size δt, given by

$$\frac{\delta S(t)}{S(t)} = \sigma\delta X(t) + \mu\delta t,$$

where $X = (X(t), t \geq 0)$ is a Lévy process, $\sigma > 0$ is the volatility parameter and μ is the stock drift parameter.

The stochastic differential equation for the price is

$$\begin{aligned} dS(t) &= \sigma S(t-)\, dX(t) + \mu S(t-)\, dt \\ &= S(t-) dZ(t), \end{aligned}$$

where $Z(t) = \sigma X(t) + \mu t$. Then $S(t) = \mathcal{E}_{Z(t)}$ is the stochastic exponential of Z.

Example 5.3 *When X is a standard Brownian motion B, the price process is given by the geometric Brownian motion*

$$S(t) = \exp\left(\sigma B(t) + \left(\mu - \frac{1}{2}\sigma^2\right) t\right).$$

In order to ensure that the stock prices are non-negative, (4.3) yields

$$\Delta X(t) > -\sigma^{-1} \quad a.s.$$

for each $t > 0$. Denote $c := -\sigma^{-1}$. We impose

$$\int_{(c,-1]\cup[1,+\infty)} x^2 \nu(dx) < \infty.$$

This means that each $X(t)$ has first and second moments, which is reasonable for stock returns.

By the Lévy-Itô decomposition, we have

$$X(t) = mt + kB(t) + \int_c^\infty x\widetilde{N}(t, dx),$$

where $k \geq 0$ and $m := b + \int_{(c,-1]\cup[1,+\infty)} x\nu(dx)$ (in terms of the previous parameters). Representing $S(t)$ as the stochastic exponential $\mathcal{E}_{Z(t)}$, we obtain from (4.5) that

$$\begin{aligned} d(\log(S(t))) &= k\sigma dB(t) + \left(m\sigma + \mu - \frac{1}{2}k^2\sigma^2\right) dt \\ &+ \int_c^\infty \log(1+\sigma x)\, \widetilde{N}(dt, dx) + \int_c^\infty (\log(1+\sigma x) - \sigma x)\, \nu(dx)\, dt. \end{aligned}$$

In applications to finance, several Lévy processes are used in order to obtain realistic dynamics: jump-diffusion processes, the variance-gamma, the normal inverse Gaussian, the CGMY process, hyperbolic processes, etc.

5.3 Change of measure

We now want to find measures Q, equivalent to the original measure P, such that the discounted stock price process $\widetilde{S}$ is a Q-martingale. Let Y be a Lévy-type stochastic integral of the form

$$dY(t) = b(t)\,dt + u(t)\,dB(t) + \int_{\mathbb{R}-\{0\}} h(t,x)\,\widetilde{N}(dt,dx).$$

Consider that e^Y is an exponential martingale (therefore, b is determined by u and h). Define Q by

$$\frac{dQ}{dP} = e^{Y(T)}.$$

By the Girsanov theorem and its generalization, we have taht

$$\begin{aligned} B_Q(t) &= B(t) - \int_0^t u(s)\,ds \text{ is a } Q\text{-Brownian motion,} \\ \widetilde{N}_Q(t,A) &= \widetilde{N}(t,A) - \nu_Q(t,A) \text{ is a } Q\text{-martingale,} \\ \nu_Q(t,A) &:= \int_0^t \int_A \left(e^{h(s,x)} - 1\right) \nu(dx)\,ds. \end{aligned}$$

$\widetilde{S}(t) = e^{-rt}S(t)$ can be written in terms of these processes by

$$\begin{aligned} d\left(\log\left(\widetilde{S}(t)\right)\right) &= k\sigma dB_Q(t) + \Bigg(m\sigma + \mu - r - \frac{1}{2}k^2\sigma^2 + k\sigma u(t) \\ &+\sigma \int_{\mathbb{R}-\{0\}} x\left(e^{h(t,x)} - 1\right)\nu(dx)\Bigg)dt + \int_c^\infty \log(1+\sigma x)\,\widetilde{N}_Q(dt,dx) \\ &+ \int_c^\infty (\log(1+\sigma x) - \sigma x)\,\nu_Q(dt,dx). \end{aligned}$$

If we put $\widetilde{S}(t) = \widetilde{S}_1(t)\,\widetilde{S}_2(t)$, then

$$\begin{aligned} d\left(\log\left(\widetilde{S}_1(t)\right)\right) &= k\sigma dB_Q(t) - \frac{1}{2}k^2\sigma^2 dt \\ &+ \int_c^\infty \log(1+\sigma x)\,\widetilde{N}_Q(dt,dx) + \int_c^\infty (\log(1+\sigma x) - \sigma x)\,\nu_Q(dt,dx). \end{aligned}$$

and

$$\begin{aligned} d\left(\log\left(\widetilde{S}_2(t)\right)\right) &= (m\sigma + \mu - r + k\sigma u(t) + \\ &+\sigma \int_{\mathbb{R}-\{0\}} x\left(e^{h(t,x)} - 1\right)\nu(dx)\Bigg)\,dt. \end{aligned}$$

Apllying Itô's formula to $\widetilde{S}_1$, we obtain

$$d\widetilde{S}_1(t) = k\sigma\widetilde{S}_1(t-)\,dB_Q(t) + \int_c^\infty \sigma\widetilde{S}_1(t-)\,x\widetilde{N}_Q(dt,dx),$$

and $\widetilde{S}_1$ is a Q-martingale. Therefore, $\widetilde{S}$ is a Q-martingale if and only if

$$m\sigma + \mu - r + k\sigma u(t) + \sigma\int_{\mathbb{R}-\{0\}} x\left(e^{h(t,x)} - 1\right)\nu(dx) = 0 \quad \text{a.s.} \tag{5.1}$$

In general, the equation (5.1) may have an infinite number of possible solution pairs (u,h). There are an infinite number of possible measures Q such that $\widetilde{S}$ is a Q-martingale. Therefore, in general, a Lévy process model is an incomplete market model.

Example 5.4 *(Brownian motion or Black-Scholes model) Let $\nu = 0$ and $k \neq 0$. Then, exists a unique solution for (5.1):*

$$u(t) = \frac{r-\mu-m\sigma}{k\sigma} \quad a.s.$$

and the market is complete (Black-Scholes model).

Example 5.5 *(Poisson process case) Take $k = 0$ and $\nu(x) = \lambda\delta_1(x)$. Then $X(t) = mt + \int_c^\infty x\widetilde{N}(t,dx)$, where the jump part is the standard Poisson process $N(t)$. Writing $h(t,1) = h(t)$, we have from (5.1) that*

$$m\sigma + \mu - r + \sigma\lambda\left(e^{h(t)} - 1\right) = 0 \quad a.s.$$

and

$$h(t) = \log\left(\frac{r-\mu+(\lambda-m)\sigma}{\lambda\sigma}\right).$$

In this case, the market is also complete and we obtain a martingale measure if $r-\mu+(\lambda-m)\sigma > 0$.

In most part of the other cases (with other Lévy processes), the market is incomplete.

5.4 Incomplete markets and Esscher transform

In Lévy market models, equivalent measures Q exist such that $\widetilde{S}$ will be a Q-martingale. However, these measures are not unique (in most cases). We must consider a selection rule or principle in order to reduce the class of all possible measures Q to an appropriate subset. Then we must apply some procedure in order to obtain a unique equivalent measure Q.

Consider the aditional assumption

$$\int_{|x|\geq 1} e^{ux}\nu(dx) < \infty,$$

for all $u \in \mathbb{R}$. We can consider the analytic continuation of the Lévy- Khintchine formula, in order to get

$$\mathbb{E}\left[e^{-uX(t)}\right] = e^{-t\psi(u)},$$

where

$$\psi(u) = -\eta(iu) = bu - \frac{1}{2}k^2u^2 + \int_c^\infty \left(1 - e^{-ux} - ux\mathbf{1}_{\{|x|<1\}}(x)\right)\nu(dx).$$

The processes

$$\begin{aligned} M_u(t) &= \exp\left(iuX(t) - t\eta(u)\right), \\ N_u(t) &= M_{iu}(t) = \exp\left(-uX(t) + t\psi(u)\right) \end{aligned}$$

are martingales and N_u is strictly positive. Define a new measure by

$$\frac{dQ_u}{dP}|_{\mathcal{F}_t} = N_u(t).$$

The measure Q_u is usually called the Esscher transform of P by the martingale N_u. Applying the Itô formula to N_u, we get

$$dN_u(t) = N_u(t-)\left(-kuB(t) + \left(e^{-ux} - 1\right)\widetilde{N}(dt, dx)\right).$$

Comparing this with (4.9) for an exponential martingale e^Y, we have that

$$\begin{aligned} u(t) &= -ku, \\ h(t,x) &= -ux \end{aligned}$$

and for Q_u to be a martingale as in (5.1), we require that

$$m\sigma + \mu - r - k^2u\sigma + \sigma\int_c^\infty x\left(e^{-ux} - 1\right)\nu(dx) = 0 \text{ a.s.}$$

Let

$$z(u) = \int_c^\infty x\left(e^{-ux} - 1\right)\nu(dx) - k^2u.$$

Then, the martingale condition is

$$z(u) = \frac{r - \mu - m\sigma}{\sigma}. \tag{5.2}$$

Since $z'(u) < 0$, z is strictly decreasing, and therefore there is a unique u (a unique measure Q_u) that satisfies (5.2).

The Esscher transform is such that the measure Q_u minimizes the relative entropy $H(Q|P)$ between the measures Q and P (a measure of "distance" between two measures), where

$$H(Q|P) = \mathbb{E}_Q \left[\ln \left(\frac{dQ}{dP} \right) \right] = \mathbb{E}_P \left[\frac{dQ}{dP} \ln \left(\frac{dQ}{dP} \right) \right].$$

5.5 Absence of arbitrage

Let X be a Lévy process and consider a market model where $S_t = S_0 \exp(X_t)$.

Theorem 5.6 *If the trajectories of X are neither increasing (a.s.) nor decreasing (a.s.), then the exponential Lévy market model given by $S_t = S_0 \exp(X_t)$ is arbitrage free: there exists a measure Q equivalent to P such that $\widetilde{S}_t = e^{-rt} S_t$ is a Q-martingale.*

In other words, the exponential-Lévy model is arbitrage free in the following cases (not mutually exclusive):

- 1) X has a nonzero Gaussian component (or diffusion coeff.): $\sigma > 0$.
- 2) X has infinite variation: $\int_{|x|<1} |x| \, \nu(dx) = \infty$.
- 3) X has both positive and negative jumps.
- 4) X has positive jumps and negative drift or negative jumps and positive drift.

5.6 The mean-correcting measure

A practical way to obtain an equivalent martingale measure Q in a exponential Lévy model of type $S_t = S_0 \exp(X_t)$, is by mean correcting the exponential of a Lévy process (see [7], pages 79-80). We can correct the exponential of the Lévy process X, by adding a new drift term mt (with new parameter m):

$$\overline{X}_t = mt + X_t.$$

When comparing the characteristics triplet of $\overline{X}$ with those of X, the only parameter that changes is the drift: $\overline{b} = b + m$. We can change the m parameter of the process X such that $\widetilde{S}_t = e^{-rt}S_t$ is a martingale. This is equivalent to choose an equivalent martingale measure Q.

Example 5.7 *In the Black-Scholes model, we change the mean of the normal distribution $\mu - \frac{1}{2}\sigma^2 = m_{old}$ into the new m parameter:*

$$m_{new} = r - \frac{1}{2}\sigma^2,$$

or

$$m_{new} = m_{old} + r - \ln\left[\varphi\left(-i\right)\right],$$

where $\varphi(x)$ is the characteristic function of the log-returns involving the m_{old} parameter. In the Black-Scholes model, $\ln\left[\varphi\left(-i\right)\right] = \mu$. This choice of m_{new} will imply that the discounted price $\widetilde{S}_t = e^{-rt}S_t$ is a martingale.

Procedure:
1) Estimate in some way the parameters involved in the process.
2) Then change the m parameter in a way that

$$m_{new} = m_{old} + r - \ln\left[\varphi\left(-i\right)\right],$$

where $\varphi(x)$ is the characteristic function of the log-returns involving the m_{old} parameter.

3) Then, with this new m_{new} parameter in the Lévy process, the discounted price $\widetilde{S}_t = e^{-rt}S_t$ is a martingale and we have chosen the mean-correcting equivalent martingale measure.

In page 78 of [7], the author lists what is the value of the m parameter for several Lévy processes (CGMY, VG, NIG, etc...)

5.7 Hyperbolic processes in finance

Let $A \in \mathcal{B}(\mathbb{R})$ be measurable set and let $(g_\theta, \theta \in A)$ be a family of probability density functions, and ρ a probability distribution on A (called mixing measure). The "probability mixture"

$$h(x) = \int_A g_\theta(x)\,\rho(d\theta)$$

is a probability density function on $\mathbb{R}$. The hyperbolic distributions are "probability mixtures".

Consider the Bessel functions of the 3rd kind:

$$K_\nu(x) = \frac{1}{2}\int_0^\infty u^{\nu-1} \exp\left(-\frac{1}{2}x\left(u+\frac{1}{u}\right)\right) du, \quad x,\nu \in \mathbb{R}.$$

For each $a, b > 0$

$$f_\nu^{a,b}(x) = \frac{\left(\frac{a}{b}\right)^{\frac{\nu}{2}}}{2K_\nu\left(\sqrt{ab}\right)} x^{\nu-1} \exp\left(-\frac{1}{2}\left(ax+\frac{b}{x}\right)\right)$$

is a pdf on $(0,\infty)$, which is called a Generalized Inverse Gaussian or $GIG(\nu,a,b)$. Take ρ to be $GIG(1,a,b)$ and $A=(0,\infty)$ and g_{σ^2} the pdf of $N(\mu+b\sigma^2,\sigma^2)$ with $\mu, b \in \mathbb{R}$. The resulting probability mixture is

$$h_{\delta,u}^{\alpha,\beta}(x) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \exp\left(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right),$$

for all $x \in \mathbb{R}$, where $\alpha^2 = a+\beta^2$ and $\delta^2 = b$. The corresponding law is called an hyperbolic distribution ($log\left(h_{\delta,u}^{\alpha,\beta}\right)$ is a hyperbola).The parameter μ is a location parameter, α is a "tail" parameter, β controls the asymmetry and δ is a scale parameter. These distributions are infinitely divisible and all their moments exist. The moment generating function is

$$M_{\delta,u}^{\alpha,\beta}(u) = \int_{\mathbb{R}} e^{ux} h_{\delta,u}^{\alpha,\beta}(x)\,dx.$$

It can be proved that

$$M_{\delta,u}^{\alpha,\beta}(u) = e^{\mu u}\frac{\sqrt{\alpha^2-\beta^2}}{K_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)}\frac{K_1\left(\delta\sqrt{\alpha^2-(\beta+u^2)}\right)}{\sqrt{\alpha^2-(\beta+u^2)}}$$

The characteristic function is

$$\varphi(u) = M(iu).$$

For simplicity, we restrict to the symmetric case ($\mu=\beta=0$) and with $\zeta = \delta\alpha$,

$$h_{\zeta,\delta}(x) = \frac{1}{2\delta K_1(\zeta)} \exp\left(-\zeta\sqrt{1+\left(\frac{x}{\delta}\right)^2}\right).$$

The corresponding Lévy process has no Gaussian part and it is

$$X_{\zeta,\delta}(t) = \int_0^t \int_{\mathbb{R}-\{0\}} x\widetilde{N}(ds,dx).$$

5.8 Option pricing with hyperbolic processes

The stock price process can be modeled by

$$dS(t) = S(t-)\, dX_{\zeta,\delta}(t).$$

The jumps of $X_{\zeta,\delta}$ are not bounded from below and this represents a major drawback for this approach. In order to solve this problem, one can model

$$S(t) = S(0)\, e^{X_{\zeta,\delta}(t)},$$
$$\widetilde{S}(t) = S(0)\, e^{X_{\zeta,\delta}(t)-rt}.$$

The martingale measure Q is such that $\widetilde{S}$ is a Q martingale. The market is incomplete and therefore we can apply the Esscher transform and consider the new measure Q_u such that

$$\frac{dQ_u}{dP}|_{\mathcal{F}_t} = N_u(t) = \exp\left(-uX_{\zeta,\delta}(t) - t\log\left(M_{\zeta,\delta}(u)\right)\right).$$

By the Generalized Girsanov theorem, $\widetilde{S}$ is a Q-martingale iff $\widetilde{S}N_u$ is a P-martingale. The process $\widetilde{S}N_u$ is given by

$$\widetilde{S}(t)\, N_u(t) = \exp\left((1-u)\, X_{\zeta,\delta}(t) - t\left(\log\left(M_{\zeta,\delta}(u)\right) + r\right)\right).$$

One can prove that

$$\exp\left((1-u)\, X_{\zeta,\delta}(t) - t\log\left(M_{\zeta,\delta}(1-u)\right)\right)$$

is a martingale. Therefore, $\widetilde{S}$ is a Q-martingale if and only if

$$r = \log\left(M_{\zeta,\delta}(1-u)\right) - \log\left(M_{\zeta,\delta}(1-u)\right) =$$
$$= \log\left[\frac{K_1\sqrt{\zeta^2 - \delta^2(1-u)^2}}{K_1\left(\sqrt{\zeta^2-\delta^2u^2}\right)}\right] - \frac{1}{2}\log\left[\frac{\zeta^2-\delta^2(1-u)^2}{\zeta^2-\delta^2u^2}\right].$$

The value of u can be obtained from the previous expression by using appropriate numerical procedures. One can now price an European call option, by using the formula

$$V(0) = \mathbb{E}_{Q_u}\left[e\left(se^{X_{\zeta,\delta}(T)} - K\right)^+\right]$$

If $f^{(t)}_{\zeta,\delta}$ is the probability density function of $X_{\zeta,\delta}(t)$ with respect to P, then we can use the Esscher transform to show that $X_{\zeta,\delta}(t)$ has the following probability density function with respect to Q_u:

$$f^{(t)}_{\zeta,\delta}(x;u) = f^{(t)}_{\zeta,\delta}(x)\, e^{-ux-t\log\left(M_{\zeta,\delta}(u)\right)}.$$

Therefore, the pricing formula is:

$$V(0) = s\int_{\log\left(\frac{k}{x}\right)}^{\infty} f_{\zeta,\delta}^{(T)}(x;1-u)\,dx - e^{-rT}K\int_{\log\left(\frac{k}{x}\right)}^{\infty} f_{\zeta,\delta}^{(T)}(x;u)\,dx.$$

- Volatility: If we had $S(t) = e^{Z(t)}$ with $Z(t) = \sigma B(t)$ (where B is a Brownian motion) then the volatility is

$$\sigma^2 = \mathbb{E}\left[Z(1)^2\right].$$

- By an analogy argument, in the hyperbolic case the volatility can be defined by

$$\sigma^2 = \mathbb{E}\left[X_{\zeta,\delta}(1)^2\right],$$

  and one can prove that (from the moment generating function and Bessel functions properties)

$$\sigma^2 = \frac{\delta^2 K_2(\zeta)}{\zeta K_1(\zeta)}.$$

Chapter 6

Risk neutral valuation and parameter estimation

6.1 Risk neutral valuation

In an arbitrage-free market modeled by an exponential Lévy process (or exponential Lévy model), the price process of the underlying risky asset is given by

$$S_t = S_0 \exp(X_t),$$

where X_t is a Lévy process. In an exponential Lévy model, the discounted price process

$$\widetilde{S}_t = e^{-rt} S_t$$

is a martingale with respect to some martingale measure (or risk neutral measure) Q.

The value $\Pi_t(H_T)$ of a contingent claim (option of derivative) with payoff H_T, is given by the risk-neutral valuation formula:

$$\Pi_t(H_T) = e^{-r(T-t)} E_Q[H_T | \mathcal{F}_t] \tag{6.1}$$

Specifying an option pricing model is equivalent to specify the law of S_t under the risk-neutral measure. In the Black-Scholes model, the dynamics of S_t under Q can be defined by

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where W_t is a standard Brownian motion under Q. Alternatively, we can define

$$S_t = S_0 \exp(X_t),$$

where

$$X_t = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t.$$

For most exponential Lévy models, it is impossible to find a closed form solution, even for plain vanilla derivatives (the Black and Scholes model is an exception). We assume that the martingale measure Q has been chosen (the mean-correcting martingale measure, for example). Assume that we know the density $f_{\mathbb{Q}}$ of S_T under the equivalent risk neutral measure Q. Then, we have for the price of an European call with strike K and maturity T, at time 0 (see Eq. (6.1)):

$$\begin{aligned} C_0 &= \exp(-rT) E_Q \left[(S_T - K)^+ \right] \\ &= \exp(-rT) \int_0^{+\infty} f_{\mathbb{Q}}(x) (x - K)^+ \, dx \\ &= \exp(-rT) \int_K^{+\infty} x f_{\mathbb{Q}}(x) \, dx - K \exp(-rT) \Pi_2, \end{aligned}$$

where Π_2 is the probability for the call option to be in the money at expiration. For most of the Lévy distributions, this integral should be calculated numerically and this calculation can be computationally very demanding. Moreover, we may not know explicitly $f_{\mathbb{Q}}$. Therefore, this method is of a limited interest in practice. The risk neutral density $f_{\mathbb{Q}}$ is rarely known. Nevertheless, we know, from the Lévy-Khintchine formula, the equation for the Fourier transform of S_t. In order to evaluate an option one then needs to invert the Fourier transform. The algorithms for the inversion of the Fourier transform are fast and optimized. The Fast Fourier transform (FFT) algorithm allows the calculation of the prices of options with different strikes in a single calculation. This method was developed by Carr and Madan in [3].

Consider an European call with underlying S_t and with strike K. Define

$$\begin{aligned} k &= \ln(K), \\ s_T &= \ln(S_T). \end{aligned}$$

Let $\Phi_T(u)$ be the characteristic function of s_T, i.e.,

$$\Phi_T(u) = E\left[e^{ius_T}\right] = \int_{-\infty}^{+\infty} e^{ius} q_T(s) \, ds, \tag{6.2}$$

where $q_T(s)$ is the density of s_T. The price of the call option at time 0 is:

$$\begin{aligned} C_0(k) &= \exp(-rT) E\left[(S_T - K)_+ \right] \\ &= \exp(-rT) \int_k^{\infty} \left(e^s - e^k\right) q_T(s) \, ds. \end{aligned} \tag{6.3}$$

The function $C_0(k)$ as a function of k is not square-integrable because as $k \to -\infty$ we have that $K \to 0$, $C_0(k) \to S_0$ and therefore $C_0(k)$ is not integrable. But $C_0(k)$ as a function of k should be square-integrable in order to calculate the inverse Fourier transform. Carr and Madam suggested to consider a "modified call price" function:

$$c_0(k) = \exp(\alpha k) C_0(k),$$

with $\alpha > 0$ in order to ensure the integrability when $k \to -\infty$.

6.2 Valuation with the Fourier transform

The Fourier transform of $c_0(k)$ is

$$\Psi_T(v) = \int_{-\infty}^{+\infty} e^{ivk} c_0(k)\, dk \tag{6.4}$$

Since

$$c_0(k) = \exp(\alpha k) C_0(k) \underset{k \to -\infty}{\approx} S_0 \exp(\alpha k),$$

this function is square integrable in $-\infty$. Inverting the Fourier transform, we obtain:

$$c_0(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \Psi_T(v)\, dv,$$

$$C_0(k) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \Psi_T(v)\, dv.$$

But $C_0(k)$ is real, and therefore:

$$\mathrm{Im}\left[\int_{-\infty}^{+\infty} e^{-ivk} \Psi_T(v)\, dv\right] = 0.$$

Let $a(v)$ and $b(v)$ be the real and imaginary parts of $\Psi_T(v)$:

$$a(v) = \int_{-\infty}^{+\infty} \cos(vk)\, c_0(k)\, dk,$$

$$b(v) = \int_{-\infty}^{+\infty} \sin(vk)\, c_0(k)\, dk.$$

Then (note that a is even and b is odd)

$$\Psi_T(-v) = a(v) - ib(v).$$

Define the functions:

$$\begin{aligned}
A(k) &= \int_{-\infty}^{0} e^{-ivk}\Psi_T(v)\,dv \\
B(k) &= 2\pi \exp(\alpha k)\, C_0(k) - A(k) \\
&= \int_{0}^{+\infty} e^{-ivk}\Psi_T(v)\,dv.
\end{aligned}$$

If we change the variable $v \to -v$, then

$$\begin{aligned}
A(k) &= \int_{+\infty}^{0} -e^{ivk}\Psi_T(-v)\,dv \\
&= \int_{0}^{+\infty} \left[\cos(vk)\, a(v) + \sin(vk) b(v) + i\left(\sin(vk) a(v) - b(v)\cos(vk)\right)\right] dv.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
B(k) &= \int_{0}^{+\infty} e^{-ivk}\Psi_T(v)\,dv \\
&= \int_{0}^{+\infty} \left[\cos(vk)\, a(v) + \sin(vk) b(v) - i\left(\sin(vk) a(v) - b(v)\cos(vk)\right)\right] dv
\end{aligned}$$

Comparing both expressions,

$$\begin{aligned}
\operatorname{Re}\left[A(k)\right] &= \operatorname{Re}\left[B(k)\right], \\
\operatorname{Im}\left[A(k)\right] &= -\operatorname{Im}\left[B(k)\right]
\end{aligned}$$

Then, it is easy to see that

$$\begin{aligned}
2\pi \exp(\alpha k)\, C_0(k) &= A(k) + B(k) \\
&= 2\operatorname{Re}\left[B(k)\right]
\end{aligned}$$

and therefore

$$C_0(k) = \frac{\exp(-\alpha k)}{\pi} \operatorname{Re}\left[\int_{0}^{+\infty} e^{-ivk}\Psi_T(v)\,dv\right]. \qquad (6.5)$$

Now, let us try to express Ψ_T as a function of Φ_T. From (6.3) and (6.4), we have

$$\Psi_T(v) = e^{-rT}\int_{-\infty}^{+\infty}\int_{k}^{+\infty} e^{\alpha k} e^{ivk}\left(e^{s} - e^{k}\right) q_T(s)\,ds dk.$$

Using the Fubini theorem and changing the order of integration, we have:

$$
\begin{aligned}
\Psi_T(v) &= e^{-rT} \int_{-\infty}^{+\infty} \int_{-\infty}^{s} \left(e^{ivk+\alpha k+s} - e^{ivk+k(\alpha+1)}\right) q_T(s)\, dk ds \\
&= e^{-rT} \int_{-\infty}^{+\infty} q_T(s) \left[\frac{e^{ivk+\alpha k+s}}{iv+\alpha} - \frac{e^{ivk+k(\alpha+1)}}{iv+\alpha+1}\right]_{-\infty}^{s} ds \\
&= e^{-rT} \int_{-\infty}^{+\infty} q_T(s) \left(\frac{e^{ivs+\alpha s+s}}{iv+\alpha} - \frac{e^{ivs+s(\alpha+1)}}{iv+\alpha+1}\right) ds \\
&= e^{-rT} \int_{-\infty}^{+\infty} q_T(s)\, e^{(iv+\alpha+1)s} \left(\frac{1}{(iv+\alpha)(iv+\alpha+1)}\right) ds \\
&= \frac{e^{-rT}}{\alpha^2+\alpha-v^2+iv(2\alpha+1)} \Phi_T(v - i(1+\alpha)),
\end{aligned} \tag{6.6}
$$

where Φ_T is the characteristic function of s_T - see eq. (6.2). We assume that $c_0(k)$ is integrable when $k \to +\infty$, i.e., we assume that

$$
\Psi_T(0) = \int_{-\infty}^{+\infty} c_0(k)\, dk < \infty.
$$

This condition in terms of Φ_T is

$$
\Phi_T(-i(1+\alpha)) < \infty
$$

or

$$
\int_{-\infty}^{+\infty} e^{(1+\alpha)s} q_T(s)\, ds < \infty,
$$

which is equivalent to

$$
\mathbb{E}\left[S_T^{1+\alpha}\right] < \infty.
$$

The final formula for the price of a call option in terms of Φ_T is (see (6.5) and (6.6))

$$
C_0(k) = \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\int_0^{+\infty} \frac{e^{-ivk}\Phi_T(v-i(1+\alpha))}{\alpha^2+\alpha-v^2+iv(2\alpha+1)} dv\right].
$$

Carr and Madan suggest to choose $\alpha \approx 0.25$. W. Schoutens proposes $\alpha \approx 0.75$. The choice of α affects the convergence speed.

6.3 The Fast Fourier Transform

In order to calculate $C_0(k)$, we discretize the integral

$$C_0(k) = \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\int_0^{+\infty} e^{-ivk} \Psi_T(v)\, dv\right]$$
$$\approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\int_0^{(N-1)\eta} e^{-ivk} \Psi_T(v)\, dv\right],$$

where η is the integration step and N is a large positive integer. Using the trapezoidal method for the integral approximation (with coefficients $\frac{1}{2}$ for the first and the last terms in the sum), we have

$$C_0(k) \approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\sum_{j=0}^{N-1} e^{-iv_j k} \Psi_T(v_j) \cdot \eta \cdot w_j\right],$$

where $v_j = \eta \cdot j$ and

$$w_j = \begin{cases} \frac{1}{2} & \text{if } j = 0 \text{ or } j = N-1, \\ 1 & \text{if } \quad 0 < j < N-1. \end{cases}$$

We should center the analysis on the options around the options at-the money: $K = S_0$ or $k = \ln(S_0) := \theta$. Therefore, define

$$k_u = \theta - b + \lambda u, \quad u = 0, ..., N-1,$$
$$\lambda = \frac{2b}{N-1}.$$

Hence

$$C_0(k_u) \approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \operatorname{Re}\left[\sum_{j=0}^{N-1} e^{-i\eta j(\theta - b + \lambda u)} \Psi_T(\eta j) \cdot \eta \cdot w_j\right]$$
$$\approx \frac{e^{-\alpha k} e^{-rT}}{\pi} \eta \operatorname{Re}\left[\sum_{j=0}^{N-1} e^{-i\eta j \lambda u} \Psi_T(\eta j) \cdot e^{i\eta j(\theta - b)} \cdot w_j\right]$$

With the Fast Fourier Transform algorithm (FFT), we can calculate the N values of the sum

$$w(u) = \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N} ju} x(j), \quad u = 0, 1, ..., N-1,$$

with a number of product operations of $N \ln(N)$ instead of N^2. In order to apply the FFT algorithm, we must choose

$$\eta\lambda = \frac{2\pi}{N}.$$

6.4 Parameter estimation

Assume that the underlying asset follows the exponential of a particular Lévy process. In order to estimate the parameters of the model, we use market data, for example the prices of european calls on some index at some fixed date. We use present values to estimate the parameter and not past or historical data.

As an example, in order to calibrate the pricing model on the data, we choose parameters of the Variance Gamma model in order to minimize the quadratic error between the market prices of the call options and the call options prices given by the model. After estimating the parameters, if we want to price path-dependent options or exotic options (for example, barrier options), we can use Monte Carlo techniques to simulate a large number of paths of the Variance Gammma process with the optimized parameters previously estimates and we can calculate the exotic options by Monte-Carlo method from the formula:

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q\left[X|\mathcal{F}_t\right].$$

The data for model calibration can be, for example, the 77 call option prices on S&P 500 Index at the close of the market on 18 April 2002 (see [7], page 155).

The characteristic function of the Variance-Gamma distribution with parameters (σ, ν, θ):

$$\Phi_{VG}(u; \sigma, \nu, \theta) = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-1/\nu}.$$

We can define the Variance-gamma process as a Lévy Process $X_t^{(VG)}$ such that the distribution of the increment $X_{t+s} - X_s$ follows the Variance-Gamma law with parameters $\left(\sigma\sqrt{t}, \nu/t, t\theta\right)$ and

$$\mathbb{E}\left[e^{iuX_t^{(VG)}}\right] = \left(1 - iu\theta\nu + \frac{1}{2}\sigma^2\nu u^2\right)^{-t/\nu}.$$

The Variance-Gamma process has the following properties:

- (1) no diffusion component and it is a pure-jump process.
- (2) it has infinite activity (infinitely many (small) jumps in any finite time interval)
- (3) it has paths of finite variation: $\int_{-1}^{1} |x|\, \nu_{VG}(dx) < \infty$.

In the following table, we can find 77 call option prices on the S&P 500 Index at the close of the market on 18 April 2002. On that day, the S&P 500 Index closed at 1124.47. We had values of $r = 1.9\%$ and $q = 1.2\%$ per year.

Strike	May 2002	June 2002	Sep. 2002	Dec. 2002	March 2003	June 2003	Dec. 2003
975			161.60	173.30			
995			144.80	157.00		182.10	
1025			120.10	133.10	146.50		
1050		84.50	100.70	114.80		143.00	171.40
1075		64.30	82.50	97.60			
1090	43.10						
1100	35.60		65.50	81.20	96.20	111.30	140.40
1110		39.50					
1120	22.90	33.50					
1125	20.20	30.70	51.00	66.90	81.70	97.00	
1130		28.00					
1135		25.60	45.50				
1140	13.30	23.20		58.90			
1150		19.10	38.10	53.90	68.30	83.30	112.80
1160		15.30					
1170		12.10					
1175		10.90	27.70	42.50	56.60		99.80
1200			19.60	33.00	46.10	60.90	
1225			13.20	24.90	36.90	49.80	
1250				18.30	29.30	41.20	66.90

Figure 6.1: The data (from [7])

- (4) it has Lévy measure

$$\nu_{VG}(dx) = \begin{cases} C\exp(Gx)\,|x|^{-1}\,dx & \text{if } x < 0, \\ C\exp(-Mx)x^{-1}dx & \text{if } x > 0, \end{cases}$$

where $C = 1/v > 0$, $G = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\theta\nu\right)^{-1} > 0$, $M = \left(\sqrt{\frac{1}{4}\theta^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\theta\nu\right)^{-1} > 0$.

Under the historical probability measure $\mathbb{P}$, assume that the price of the risky asset is

$$S_t = S_0 \exp\left(m_H t + X_t^{VG}(\sigma_H, \nu_H, \theta_H) + w_H t\right),$$

where by the subscripts H we mean that these parameters are the ones under the historical probability measure $\mathbb{P}$. The parameter w_H is chosen such that it cancels the drift of the process $X_t^{VG}(\sigma_H, \nu_H, \theta_H)$ and therefore

$$w_H = \frac{1}{\nu_H}\ln\left(1 - \theta_H\nu_H - \frac{\sigma_H^2\nu_H}{2}\right)$$

and m_H is the expected rate of return under $\mathbb{P}$.

In order to price, we choose to estimate the parameters not under $\mathbb{P}$ but under $\mathbb{Q}$ (the risk neutral (RN) measure or equivalent martingale measure). Under $\mathbb{Q}$, the price process is

$$S_t = S_0 \exp\left(rt + X_t^{VG}(\sigma_{RN}, \nu_{RN}, \theta_{RN}) + w_{RN} t\right).$$

The parameter w_{RN} is chosen such that the discounted price process $\widetilde{S}_t = e^{-rt}S_t$ is a $\mathbb{Q}$ martingale (mean correcting equivalent martingale measure) and this results in

$$w_{RN} = \frac{1}{\nu_{RN}}\ln\left(1 - \theta_{RN}\nu_{RN} - \frac{\sigma_{RN}^2\nu_{RN}}{2}\right).$$

In practice, we need to calculate the characteristic function at the point $1/i$.

The algorithm is:

- For a set of market prices of N calls, we choose the risk neutral parameters such that the quadratic error between market prices and the prices given by the model of the call options is minimum, and is given by the root-mean-square error

$$RMSE = \min_{\sigma_{RN},\nu_{RN},\theta_{RN}} \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[(\text{market price})_i - (\text{calculated price})_i\right]^2}.$$

- Call prices are calculated by the Fourier Transform method with the FFT algorithm
- The grid of the logarithm of the strike is such that allows to interpolate with an acceptable error the prices of options for the strikes which are really traded on the market.

We now present some results obtained by Schoutens and described in [7] (page 81), for the calibration procedure whith the CGMY model:

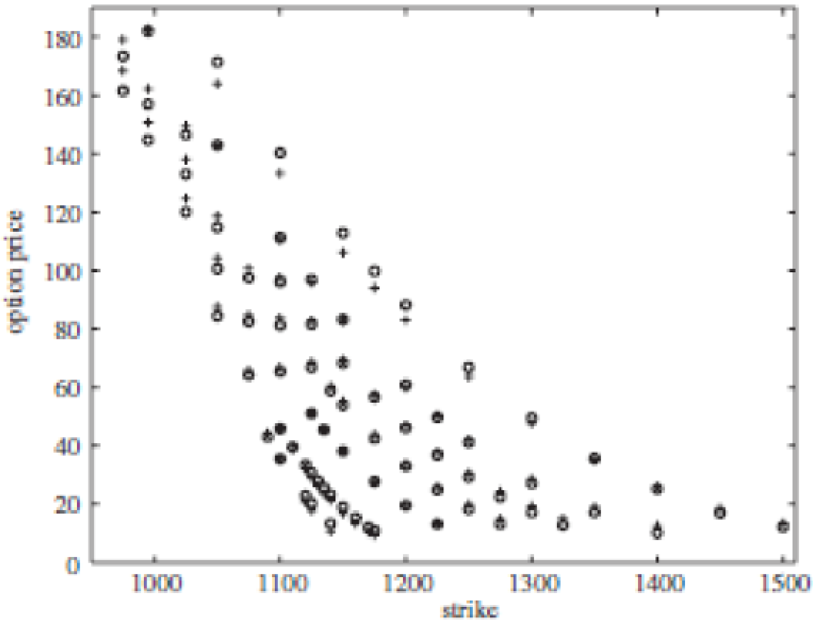


Figure 6.3 CGMY (mean-correcting) calibration of S&P 500 options (circles are market prices, pluses are model prices).

Calibration from [7]

From [7] (page 83), we obtain the comparison of the calibration results for several models.

6.5 Exotic option pricing

The payoff of an "Up and In call option" (barrier option) with strike K and barrier H is equal to the payoff of the european call, if the underlying reached or crossed the barrier H between time 0 and T. If the barrier has not been reached, then the payoff is 0.

Table 6.4 Lévy models (mean-correcting): APE, AAE, RMSE, ARPE.

Model	APE (%)	AAE	RMSE	ARPE (%)
BS	8.87	5.4868	6.7335	16.92
CGMY	3.38	2.0880	2.7560	4.96
GH	3.60	2.2282	2.8808	5.46
VG	4.67	2.8862	3.5600	7.56
NIG	3.97	2.4568	3.1119	6.17
Meixner	4.19	2.5911	3.2451	6.71

Figure 6.2: Table from [7]

After estimating the parameters, assume that we want to price an exotic option (for example, a barrier option of the type "Up and in"). We can use the Monte-Carlo method from the formula:

$$V(0) = e^{-rT}\mathbb{E}_Q\left[(S_T - K)^+ \mathbf{1}_{\{\max(S_t;0\leq t\leq T)\geq H\}}(\omega)\right],$$

where H is the barrier level. Note that if $H \leq K$, the up and in call and the european call with strike K and maturity T have the same value, because if $S_T > K$ then $S_T > H$ also.

The Monte Carlo algorithm:

- (1) We assume that the parameters of the risk neutral process were previously calibrated on the market prices of european calls by the method previously described
- (2) A large number N of trajectories of the risk neutral process is simulated on a regular time grid.
- (3) For each trajectory i $(i = 1, 2, ..., N)$ we calculate the payoff of the option by formula:

$$C_i = \left[(S_T - K)^+ \mathbf{1}_{\{\max(S_t;0\leq t\leq T)\geq H\}}(\omega_i)\right]$$

- (4) The final price of the option can be estimated by the discounted mean of the payoff for the N trajectories:

$$\widehat{V}(0) = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} C_i.$$

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