Some supplementary problems - part 1

Lévy Processes and Applications

1. Consider the Lévy measure

$$\nu(x) = \frac{1}{x^{\frac{3}{2}}} \mathbf{1}_{\{x>0\}}(x)$$

associated to the Lévy process $X(t)$.

(a) Calculate

$$\mathbb{E}\left[\int_{\varepsilon}^{1} x^2 N(t, dx)\right], \quad \text{and}$$
$$\text{Var}\left[\int_{\varepsilon}^{1} x^2 N(t, dx)\right].$$

(b) Show that

$$\sum_{\{0 \le s \le t : \Delta X(s) \in [\varepsilon, 1]\}} (\Delta X(s))^2 - \frac{2}{3} t \left(1 - \varepsilon^{\frac{3}{2}}\right)$$

is a martingale .

2. Consider the Poisson integral

$$\int_{1}^{+\infty} x^{\frac{1}{4}} N(t, dx)$$

and the associated Lévy measure $\nu(x) = \frac{1}{x^{\frac{7}{4}}} \mathbf{1}_{\{x>0\}}(x)$. Calculate $\text{Var}\left[\int_{1}^{+\infty} x^{\frac{1}{4}} N(t, dx)\right]$.

3. Consider a simple process

$$F(t, x) = \sum_{j=1}^{m} \sum_{k=1}^{n} F_k(t_j) \mathbf{1}_{(t_j, t_{j+1}]}(t) \mathbf{1}_{A_k}(x),$$

where the sets A_k are disjoint for different values of k and $t \in [0, T]$. Show that

$$\mathbb{E}\left[(I(F))^2\right] = \int_0^T \int_E \mathbb{E}\left[|F(t, x)|^2\right] \nu(dx)\, dt.$$

4. Let X be a Lévy-type of integral of the form

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dB(t) + \int_{|x|>1} \gamma(t, x) N(dt, dx) + \int_{|x| \le 1} \gamma(t, x) \widetilde{N}(dt, dx),$$

with $\mu(t) + \frac{(\sigma(t))^2}{2} + \int_{\mathbb{R}} \left(e^{\gamma(t,x)} - 1 - \gamma(t,x)\,\mathbf{1}_{\{|x|\leq 1\}}(x)\right)\nu(dx) = 0$ a.s. for all t.

(a) Show that

$$e^{X(t)} = e^{X_0} + \int_0^t F(s)dB(s) + \int_0^t \int_{\mathbb{R}} H(s,x)\,\widetilde{N}(ds,dx)$$

and find expressions for the processes $F(s)$ and $H(s,x)$.

(b) Assume that $|\sigma(t)|$ and $\left|\int_{\mathbb{R}} \left(e^{\gamma(t,x)} - 1\right)^2 \nu(dx)\right|$ are bounded by a constant C. Use Gronwall's Lemma in order to show that $e^{X(t)}$ is a square-integrable martingale.

(Note: Gronwall's Lemma: Let ϕ be a positive and locally bounded function on $\mathbb{R}_0^+$ such that $\phi(t) \leq a + b\int_0^t \phi(s)\,ds$ for all t, with $a, b \geq 0$. Then $\phi(t) \leq ae^{bt}$.)

5. Consider the stochastic differential equation (of the so-called Geometric Lévy process):

$$dX(t) = X(t-)\left[bdt + \sigma dB(t) + \int_{|x|<1} H(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K(t,x)\,N(dt,dx)\right],$$

where b, σ are constants, $H(t,x) \geq -1$ for all t and x and $H(t,x)$ and $K(t,x)$ are processes such that the Poisson integrals above are well defined.

Determine the solution of this equation

6. Let

$$dX_i(t) = \alpha_i(t)dt + \sigma_i(t)dB(t) + \int_{|x|<1} H_i(t,x)\,\widetilde{N}(dt,dx) + \int_{|x|\geq 1} K_i(t,x)\,N(dt,dx),$$

for $i = 1, 2$.

(a) Calculate $d(X_1(t)\,X_2(t))$.

(b) Knowing that $X(t)Y(t) = X(0)Y(0) + \int_0^t X(s)dY(s) + \int_0^t Y(s)dX(s) + [X,Y](t)$, and from the result obtained in (a), calculate $[X_1, X_2](t)$.

7. Let $X(t)$ be a Lévy-type of integral of the form

$$dX(t) = \alpha(t)\,dt + \sigma(t)dB(t) + \int_{\mathbb{R}\setminus\{0\}} \gamma(t,x)\,\widetilde{N}(dt,dx).$$

Consider the process $Y(t) = f(X(t))$ and determine the processes $a(t)$, $b(t)$ and $c(t,x)$ such that

$$dX(t) = a(t)\,dt + b(t)dB(t) + \int_{\mathbb{R}\setminus\{0\}} c(t,x)\,\widetilde{N}(dt,dx),$$

in the following cases:

(a) $Y(t) = (X(t))^2$

(b) $Y(t) = \cos(X(t))$