Lévy processes and applications - Lévy Processes

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Lévy Processes

Definition

Let $X = (X(t); t \geq 0)$ be a stochastic process. We say that X has independent increments if for each $n \in \mathbb{N}$ and each sequence $0 \leq t_1 < t_2 < \ldots < t_{n+1} < \infty$, the random variables $(X(t_{j+1}) - X(t_j); 1 \leq j \leq n)$ are independent and X has stationary increments if $X(t_{j+1}) - X(t_j) \overset{d}{=} X(t_{j+1} - t_j) - X(0)$.

Definition

We say that X is a Lévy process if
(1) $X(0) = 0$ (a.s),
(2) X has independent and stationary increments,
(3) X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Lévy Processes

- Conditions (1) and (2) imply that (3) is equivalent to $\lim_{t \searrow 0} P(|X(t)| > a) = 0$.
- The sample paths (trajectories) X are the maps $t \to X(t)(\omega)$ from $\mathbb{R}^+$ to $\mathbb{R}^d$ for each $\omega \in \Omega$.

Proposition

If X is a Levy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.

Proof: For each $n \in \mathbb{N}$, $X(t) = Y_1^{(n)}(t) + \cdots + Y_n^{(n)}(t)$, where $Y_j^{(n)}(t) = X\left(\frac{jt}{n}\right) - X\left(\frac{(j-1)t}{n}\right)$. By condition (2), these $Y_j^{(n)}(t)'s$ are iid r.v. and therefore, $X(t)$ is infinitely divisible. ■

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Theorem

If X is a Lévy process, then

$$\phi_{X(t)}(u) = e^{t\eta(u)},$$

for each $u \in \mathbb{R}^d$, where η is the characteristic exponent (or Lévy symbol) of $X(1)$.

Proof: Define $\phi_u(t) = \phi_{X(t)}(u)$. Then by condition (2), $\phi_u(t+s) = E\left[e^{i(u,X(t+s)-X(s)+X(s))}\right] = E\left[e^{i(u,X(t+s)-X(s))}\right] E\left[e^{i(u,X(s))}\right] = \phi_u(t)\,\phi_u(s)$.
On the other hand, by cond. (1), $\phi_u(0) = 1$. The map $t \to \phi_u(t)$ is clearly continuous.
The unique continuous function that satisfies all these conditions is of the form $\phi_u(t) = e^{t\alpha(u)}$.
But $X(1)$ is also infin. divis. and therefore $\phi_u(t) = e^{t\eta(u)}$ and $\alpha(u) = \eta(u)$. ■

L-K formula for Lévy Processes

- Exercise: Prove that if X is stochastically continuous , then the map $t \to \phi_{X(t)}(u)$ is continuous for each $u \in \mathbb{R}^d$ (Hint: see Applebaum, pages 43-44).
- L-K formula for a Lévy Process $X = (X(t); t \geq 0)$:

$$\phi_{X(t)}(u) = E\left[e^{i(u,X(t))}\right] = \exp\left\{t\left[i(b,u) - \frac{1}{2}(u,Au) + \right.\right.$$
$$\left.\left. + \int_{\mathbb{R}^d - \{0\}} \left[e^{i(u,y)} - 1 - i(u,y)\chi_{\widehat{B}}(y)\right]\nu(dy)\right]\right\}, \qquad (1)$$

  for each $t \geq 0$ and $u \in \mathbb{R}^d$. The characteristics (b, A, ν) are the characteristics of $X(1)$.
- Exercise: Show that if X and Y are stochastically continuous processes , so is their sum $X + Y$ (hint: use the elementary inequality: $P(|A+B| > C) \leq P\left(|A| > \frac{C}{2}\right) + P\left(|B| > \frac{C}{2}\right)$ with A, B random variables.

Lévy processes - Brownian motion

- A standard Brownian motion in $\mathbb{R}^d$ is a Lévy process B for which
  (1) $B(t) \sim N(0, tI)$.
  (2) B has continuous sample paths.
- From (1) we obtain

$$\phi_{B(t)}(u) = \exp\left\{-\frac{1}{2}t|u|^2\right\}.$$

- Main properties of standard Brownian motion (with $d = 1$):
- Brownian motion is locally Hölder continuous with exponent α for every $0 < \alpha < \frac{1}{2}$:

$$|B(t)(\omega) - B(t)(\omega)| \leq K(T,\omega)|t-s|^{\alpha},$$

  for all $0 \leq s < t \leq T$.

Lévy processes - Brownian motion

- The sample paths (trajectories) $t \to B(t)(\omega)$ are a.s. nowhere differentiable.
- For any sequence $(t_n, n \in \mathbb{N})$ with $t_n \nearrow \infty$, we have

$$\liminf_{n\to\infty} B(t_n) = -\infty \quad \text{a.s.}$$
$$\limsup_{n\to\infty} B(t_n) = +\infty \quad \text{a.s.}$$

- Law of iterated logarithm:

$$\limsup_{t\searrow 0} \frac{B(t)}{\left(2t\log\left(\log\left(\frac{1}{t}\right)\right)\right)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

Lévy processes - Brownian motion

- Simulated path of standard Brownian motion:

Lévy processes - Brownian motion

- Law of the iterated logarithm:

$$\limsup_{t\to\infty} \frac{B(t)}{(2t\log(\log t))^{\frac{1}{2}}} = 1, \quad \liminf_{t\to\infty} \frac{B(t)}{(2t\log(\log t))^{\frac{1}{2}}} = -1 \text{ a.s.}$$

Lévy Processes

Lévy processes - Brownian motion

- Given a non-negative definite symmetric $d \times d$ matrix, let σ be the square root of A (in the sense: $\sigma\sigma^T = A$) with σ a $d \times m$ matrix. Let $b \in \mathbb{R}^d$ and let B be a standard Brownian motion in $\mathbb{R}^m$.
- The process C defined by

$$C(t) = bt + \sigma B(t) \tag{2}$$

  is a Lévy process that satisfies $C(t) \sim N(tb, tA)$. Moreover, C is also a Gaussian process (all finite dimensional distributions are Gaussian).
- The process C is called Brownian motion with drift. The characteristic exponent (or Lévy symbol) of C is

$$\eta_C(u) = i(b, u) - \frac{1}{2}(u, Au).$$

- A Lévy process has continuous sample paths if and only if it is of the form (2).

Lévy processes - Poisson Process

- $N(t) \sim Po(\lambda t)$ is a process taking values in $\mathbb{N}_0$:

$$P[N(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

- Let us define the non-negative r.v. $\{T(n), n \in \mathbb{N}_0\}$ (waiting times), $T(0) = 0$,

$$T(n) = \inf\{t \geq 0 : N(t) = n\}.$$

The r.v. $T(n)$ has a gamma distribution and the inter-arrival times $T(n) - T(n-1)$ are iid with exponential distribution (with mean $1/\lambda$).

- Compensated Poisson process: $\widetilde{N} = \left(\widetilde{N}(t), t \geq 0\right)$ where $\widetilde{N}(t) = N(t) - \lambda t$. Note: $E\left[\widetilde{N}(t)\right] = 0$ and $E\left[\left(\widetilde{N}(t)\right)^2\right] = \lambda t$.

Lévy Processes

Lévy processes - Poisson Process

Lévy processes - Compound Poisson Process

- Sequence of iid r.v. $\{Z(n), n \in \mathbb{N}\}$ with values in $\mathbb{R}^d$ with law μ_Z. Let N be a Poisson process with intensity λ and independent of the $Z(n)'$ s.
- Compound Poisson process

$$Y(t) = \sum_{n=1}^{N(t)} Z(n),$$

  and $Y(t) \sim \pi(\lambda t, \mu_Z)$.
- The characteristic exponent is

$$\eta_Y(u) = \int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy).$$

- The sample paths of Y are piecewise constant with jumps at times $T(n)$, but now the jump sizes are random and the jump at $T(n)$ can be any value in the range of the r.v. $Z(n)$.

Lévy processes - Compound Poisson Process

Lévy processes - Interlacing processes

- Let C be A gaussian Lévy process and Y be a compound Poisson process (independent of C). Define

$$X(t) = C(t) + Y(t).$$

- X is a Lévy process with Lévy characteristic exponent

$$\eta_X(u) = i(m, u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d} \left(e^{i(u,y)} - 1\right) \lambda \mu_Z(dy).$$

- Let T_n represent the time of jump n. We have (interlacing process):

$$X(t) = \begin{cases} C(t) \text{ for } 0 \leq t < T_1, \\ C(T_1) + Z_1 \text{ for } t = T_1, \\ X(T_1) + C(t) - C(T_1) \text{ for } T_1 \leq t < T_2, \\ X(T_2-) + Z_2 \text{ for } t = T_2, \\ etc... \end{cases}$$

Lévy processes - Stable Lévy processes

- A stable Lévy process is a Lévy process X with characteristic exponent ($\sigma > 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$) (each $X(t)$ is a stable random variable):

Theorem

1. *when* $\alpha = 2$,

$$\eta_X(u) = i\mu u - \frac{1}{2}\sigma^2 u^2;$$

2. *when* $\alpha \neq 1, 2$

$$\eta_X(u) = i\mu u - \sigma^\alpha |u|^\alpha \left[1 - i\beta \operatorname{sgn}(u) \tan\left(\frac{\pi\alpha}{2}\right)\right]$$

3. *when* $\alpha = 1$,

$$\eta_X(u) = i\mu u - \sigma |u| \left[1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log(|u|)\right]$$

Lévy processes - Stable Lévy processes

- Important case (rotationally invariant stable Lévy processes):

$$\eta_X(u) = -\sigma^\alpha |u|^\alpha, \quad 0 < \alpha \leq 2.$$

- Why are these process important? they are self-similar!
- A process $Y = (Y(t), t \geq 0)$ is self-similar with Hurst index $H > 0$ if $(Y(at), t \geq 0)$ and $\left(a^H Y(t), t \geq 0\right)$ have the same finite dimensional distributions for all $a \geq 0$.
- By examining the characteristic functions, we can prove that a rotationally invariant stable Lévy process is self-similar with $H = 1/\alpha$.
- It can be proved that a Lévy process X is self-similar if and only if each $X(t)$ is strictly stable.

Lévy processes - Subordinators

- A subordinator is a one-dimensional Lévy process wich is increasing a.s.
- Subordinator≈ random model of time evolution: If $T = (T(t), t \geq 0)$ is a subordinator then $T(t) \geq 0$ a.s. and $T(t_1) \leq T(t_2)$ a.s. if $t_1 \leq t_2$.

Theorem

If T is a subordinator then its charact. exponent has the form

$$\eta_T(u) = i(b, u) + \int_{(0,\infty)} \left(e^{iuy} - 1\right) \lambda(dy), \tag{3}$$

where $b \geq 0$, and the Lévy measure λ satisfies: $\lambda(-\infty, 0) = 0$ and $\int_{(0,\infty)} (y \wedge 1) \lambda(dy) < \infty$.
Conversely, any mapping $\eta : \mathbb{R} \to \mathbb{C}$ of the form (3) is the charact. exponent of a subordinator.

- (b, λ) are called the characteristics of the subordinator T.

Lévy processes - Subordinators

- For each $t \geq 0$, the map $u \to E\left[e^{iuT(t)}\right]$ can be analytically continued to the region $\{iu, u > 0\}$ and we obtain (Laplace transform of the distribution):

$$E\left[e^{-uT(t)}\right] = e^{-t\psi(u)},$$

where

$$\psi(u) = -\eta(iu) = bu + \int_{(0,\infty)} \left(1 - e^{-yu}\right) \lambda(dy). \tag{4}$$

- ψ is called the Laplace exponent of the distribution.

Subordinators - Poisson case

- Poisson processes are subordinators

- Compound Poisson processes are subordinators if and only if the $Z(n)' s$ are positive r.v.

Subordinators -stable subordinators

- It can be proved (using the usual calculus) that (for $0 < \alpha < 1$ and $u \geq 0$)

$$u^{\alpha} = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{\infty} \left(1 - e^{-ux}\right) \frac{dx}{x^{1+\alpha}}.$$

- By (4) and the characteristics of a stable Lévy process, there exists an α-stable subordinator with Laplace exponent $\psi(u) = u^{\alpha}$ and the characteristics of T are $(0, \lambda)$, where $\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$.
- When we analytically continue this in order to obtain the Lévy charac. exponent, we obtain $\mu = 0$, $\beta = 1$ and $\sigma^{\alpha} = \cos(\alpha\pi/2)$.
- Exercise: Show that there exists an α-stable subordinator with Laplace exponent $\psi(u) = u^{\alpha}$ and the characteristics of T are $(0, \lambda)$, where $\lambda(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dx}{x^{1+\alpha}}$.

Subordinators -the Lévy subordinator

- The $\left(\frac{1}{2}\right)$-stable subordinator has a density given by the Lévy distribution (with $\mu = 0$ and $\sigma = \frac{t^2}{2}$):

$$f_{T(t)}(s) = \left(\frac{t}{2\sqrt{\pi}}\right) s^{-\frac{3}{2}} \exp\left(\frac{-t^2}{4s}\right).$$

- It is possible to show directly that

$$E\left[e^{-uT(t)}\right] = \int_0^{\infty} e^{-us} f_{T(t)}(s)\, ds = e^{-tu^{\frac{1}{2}}}.$$

- Exercise: Show that $E\left[e^{-uT(t)}\right] = \int_0^{\infty} e^{-us} f_{T(t)}(s)\, ds = e^{-tu^{\frac{1}{2}}}$. (Hint: Differentiate $g_t(u) = \int_0^{\infty} e^{-us} f_{T(t)}(s)\, ds$ with respect to u and make the substitution $x = \frac{t^2}{4us}$ to obtain the differential equation $g_t'(u) = -\frac{t}{2\sqrt{u}} g_t(u)$. Via the substitution $y = \frac{t}{2\sqrt{s}}$ we see that $g_t(0) = 1$ and the result follows).
- This subordinator can be represented by a hitting time of the Bm:

$$T(t) = \inf\left\{s > 0 : B(s) = \frac{t}{\sqrt{2}}\right\}. \tag{5}$$

Inverse Gaussian subordinators

- we can generalize the Lévy subordinator by replacing the Brownian motion in the hitting time by the Gaussian process $C(t) = B(t) + \mu t$ and the inverse Gaussian subordinator is:

$$T_\delta(t) = \inf \{s > 0 : C(s) = \delta t\}$$

where $\delta > 0$.
- Note: $t \to T_\delta(t)$ is the generalized inverse of a Gaussian process, in the sense that the Gaussian describes a Brownian Motion's level at a fixed time and the inverse Gaussian describes the distribution of the time a Brownian Motion with positive drift takes to reach a fixed positive level.

Lévy Processes

Inverse Gaussian subordinators

- Using martingale methods, it is possible to show that for each $t, u > 0$,

$$E\left[e^{-uT_\delta(t)}\right] = \exp\left(-t\delta\sqrt{2u+\mu^2} - \mu\right)$$

and $T(t)$ has a density:

$$f_{T_\delta(t)}(s) = \frac{\delta t}{\sqrt{2\pi}} e^{\delta t \mu} s^{-\frac{3}{2}} \exp\left[-\frac{1}{2}\left(t^2\delta^2 s^{-1} + \mu^2 s\right)\right],$$

for $s, t \geq 0$.
- In general, a r.v. with density $f_{T_\delta(1)}$ is called an inverse Gaussian and denoted by $IG(\delta, \mu)$

Gamma subordinators

- Let $T(t)$ be a Gamma process with parameters $a, b > 0$ such that $T(t)$ has a density
$$f_{T(t)}(x) = \frac{b^{at}}{\Gamma(at)} x^{at-1} e^{-bx}, \quad x \geq 0.$$
- Using some calculus, we can show that
$$\int_0^\infty e^{-ux} f_{T(t)}(x)\, dx = \left(1 + \frac{u}{b}\right)^{-at} = \exp\left(-ta \log\left(1 + \frac{u}{b}\right)\right)$$
$$= \exp\left(-t \int_0^\infty \left(1 - e^{-ux}\right) a x^{-1} e^{-bx} dx\right).$$
- Therefore, by (4), $T(t)$ is a subordinator with $b = 0$ and $\lambda(dx) = a x^{-1} e^{-bx} dx$

Simulation of a Gamma subordinator

Time change

- Important application of subordinators: time change!
- Let X be a Lévy process and let T be a subordinator independent of X. Let
$$Z(t) = X(T(t)).$$

Theorem

Z is a Lévy process

Proof: see Applebaum, pags. 56-58

Proposition

$$\eta_Z = -\psi_T \circ (-\eta_X).$$

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Time Change

Proof: Let $p_{T(t)}$ be the distribution associated to $T(t)$. Then

$$\begin{aligned} E\left[e^{t\eta_{Z(t)}(u)}\right] &= E\left(e^{i(u,Z(t))}\right) = E\left(e^{i(u,X(T(t)))}\right) \\ &= \int E\left(e^{i(u,X(T(s)))}\right) p_{T(t)}(ds) \\ &= \int e^{s\eta_X(u)} p_{T(t)}(ds) \\ &= E\left[e^{-(-\eta_X(u))T(t)}\right] \\ &= e^{-t\psi_T(-\eta_X(u))}. \qquad \blacksquare \end{aligned}$$

Brownian motion and 2 alpha stable motion

- Let T be an α-stable subordinator (with $0 < \alpha < 1$) and X be a Brownian motion with covariance $A = 2I$, independent of T. Then

$$\psi_T(s) = s^{\alpha}, \quad \eta_X(u) = -|u|^2$$

and therefore, by the Proposition,

$$\eta_Z(u) = -|u|^{2\alpha}$$

and Z is a 2α stable process.
- If $d = 1$ and T is the Lévy subordinator, then Z is the Cauchy process and each $Z(t)$ has a symmetric Cauchy distribution with $\mu = 0$ and $\sigma = 1$.
- Moreover, by (5), the Cauchy process can be constructed from two indepedent Brownian motions.

The variance gamma process

- Let $Z(t) = B(T(t))$, where T is a gamma subordinator and B is a Brownian motion. Then, the Lévy process Z is called a variance-gamma process.
- we replace the variance of B by a gamma r.v.
- Then, we have

$$\Phi_{Z(t)}(u) = E\left[e^{uiZ(t)}\right] = \left(1 + \frac{u^2}{2b}\right)^{-at},$$

where a and b are the usual parameters determining the gamma process.
- Exercise: Prove this result.

The variance gamma process

- Manipulating characteristic functions, it is possible to show that:

$$Z(t) = G(t) - L(t)$$

where G and L are independent gamma subordinators with parameters $\sqrt{2b}$ and a (difference of independent "gains" and "losses").

- From this representation, it is possible to show that $Z(t)$ has a Lévy density:

$$g_{\nu}(x) = \frac{a}{|x|^{1}} \left(e^{\sqrt{2b}x} \chi_{(-\infty,0)}(x) + e^{-\sqrt{2b}x} \chi_{(0,\infty)}(x) \right),$$

$$a > 0.$$

CGMY model

- The CGMY model (Carr, Geman, Madan and Yor) is a generalization of the variance gamma process, with Lévy density:

$$g_{\nu}(x) = \frac{a}{|x|^{1+\alpha}} \left(e^{b_1 x} \chi_{(-\infty,0)}(x) + e^{-b_2 x} \chi_{(0,\infty)}(x) \right),$$

$$a > 0,\ 0 \leq \alpha < 2, \quad b_1, b_2 \geq 0.$$

- When $b_1 = b_2 = 0$, we obtain stable Lévy processes.
- The exponential dampens the effects of large jumps.

The normal inverse Gaussian process

- Let $Z(t) = C(T(t)) + \mu t$ where $C(t) = B(t) + \beta t$ and T is an inverse Gaussian subordinator. Let α be such that $\alpha^2 \geq \beta^2$. Then Z depends on 4 parameters and has characteristic function ($\delta > 0$):

$$\Phi_{Z(t)}(\alpha, \beta, \delta, \mu)(u) = \exp\left[\delta t\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right) + i\mu tu\right]$$

- $Z(t)$ has a density

$$f_{Z(t)}(x) = C(\alpha, \beta, \delta, \mu; t)\, q\left(\frac{x - \mu t}{\delta t}\right)^{-1} K_1\left(\delta t \alpha q\left(\frac{x - \mu t}{\delta t}\right)\right) e^{\beta x},$$

where $q(x) = \sqrt{1 + x^2}$, $C(\alpha, \beta, \delta, \mu; t) = \pi^{-1}\alpha e^{\delta t\sqrt{\alpha^2 - \beta^2} - \beta\mu t}$ and K_1 is a Bessel function of the third kind.

- Applebaum, D. (2004). Lévy Processes and Stochastic Caculus. Cambridge University Press. - (Overview and chapter 1)
- Applebaum, D. (2005). Lectures on Lévy Processes, Stochastic Calculus and Financial Applications, Ovronnaz September 2005, Lecture 1 in http://www.applebaum.staff.shef.ac.uk/ovron1.pdf
- Papantaloleon, A. An Introduction to Lévy Processes with Applications in Finance. arXiv:0804.0482v2.
- Sato, K. (1999). Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press.