Interest Rate & Credit Risk Models

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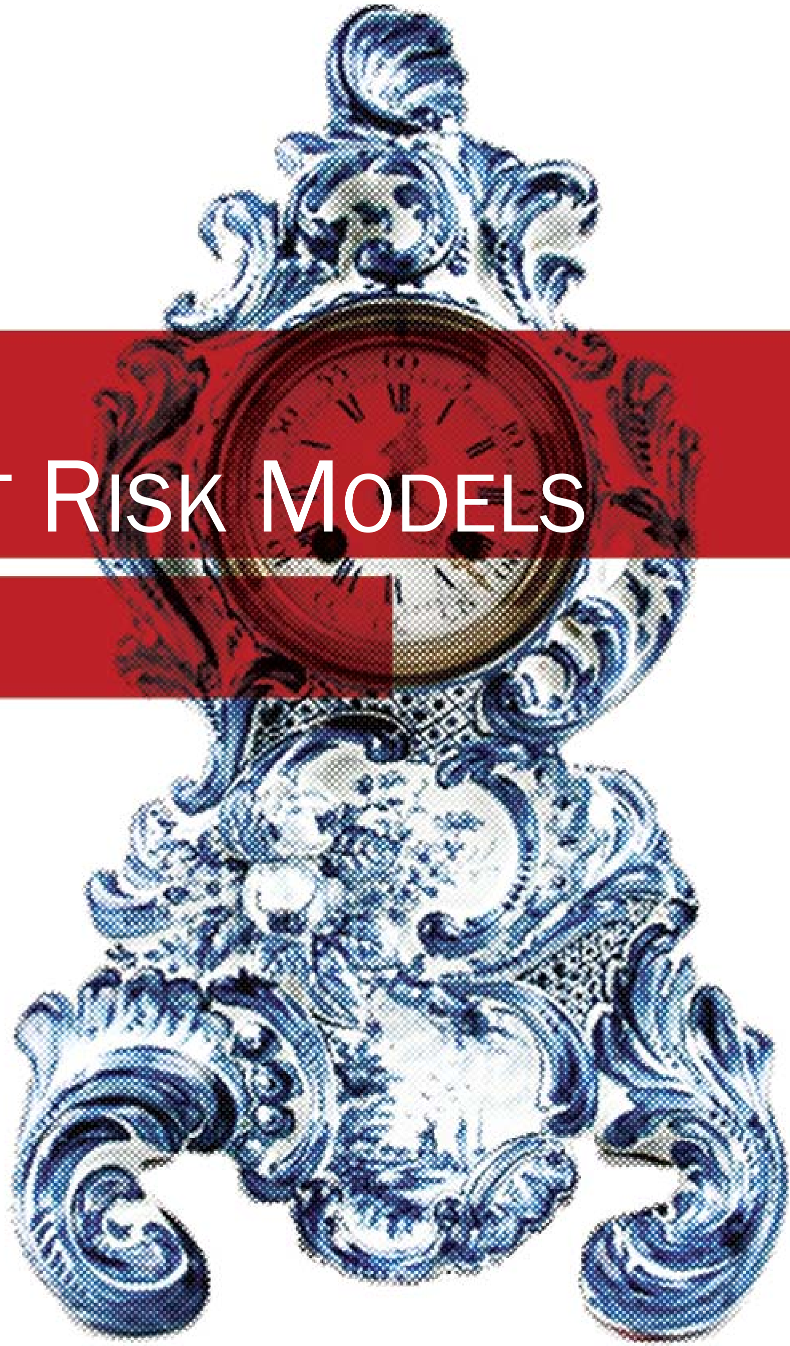
Oct 2017



LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

OBJECTIVES

Main Goal

- Be able to evaluate and manage credit and interest rate risk in increasingly complex debt markets.

Interest Rate Theory

- Understand the distinction between different interest rates in the market: spot rates, forward rates, yield-to-maturity, etc.
- Identify under which conditions a stochastic interest rate model is needed and when a deterministic interest rate models is sufficient.
- Identify and apply the stochastic spot rate models, deriving the basic term structure properties from the spot rate.
- Handle Heath-Jarrow-Morton type models for forward rates and understanding the fundamental difference the spot rate and the forward rate modeling approach.

OBJECTIVES

Credit Risk Theory

- Understand the approach of the structural models, in particular the Merton Model.
- Be familiar with main reduced form models
- Be able to price defaultable payoffs using reduced form as well as structural models
- Understand the limitations of structural and reduced form models when used to model portfolio credit derivatives
- Be able to model correlated defaults by using copulas.

PART I

INTRODUCTION TO FIXED INCOME MARKETS AND INTEREST RATE RISK

FIXED INCOME MARKETS

- Conceptually, the fixed-income market is considered as the global financial market on which various fixed interest rate instruments, such as bonds, swaps, FRAs, swaptions and caps are traded.

- In the real world, several fixed-income markets operate and consequently many concepts of interest rates have been developed.

- Interest Rate Risk - changes in the net present value (the price) of a stream of future cash flows resulting from changes in interest rates.

- Management of interest rate risk - pricing and hedging of interest rate products and balance sheets.

INTEREST RATE RISK IN BANKS BALANCE SHEETS

- 2 types of interest rate risk:
  - Risk of Net Interest Income fluctuation
  - Risk of optionality embedded in assets and liabilities, e.g. prepayment of loans and early redemption of deposits, impacting on cash-flows.
- Target variables:
  - Net Interest Income: captures cash-flows in a given period (e.g. 1 year)
  - NPV of assets minus liabilities: allows to capture all balance sheet cash-flows, usually through duration.
- Earnings-at-risk (EaR):
  - Calculated from several scenarios for interest rates

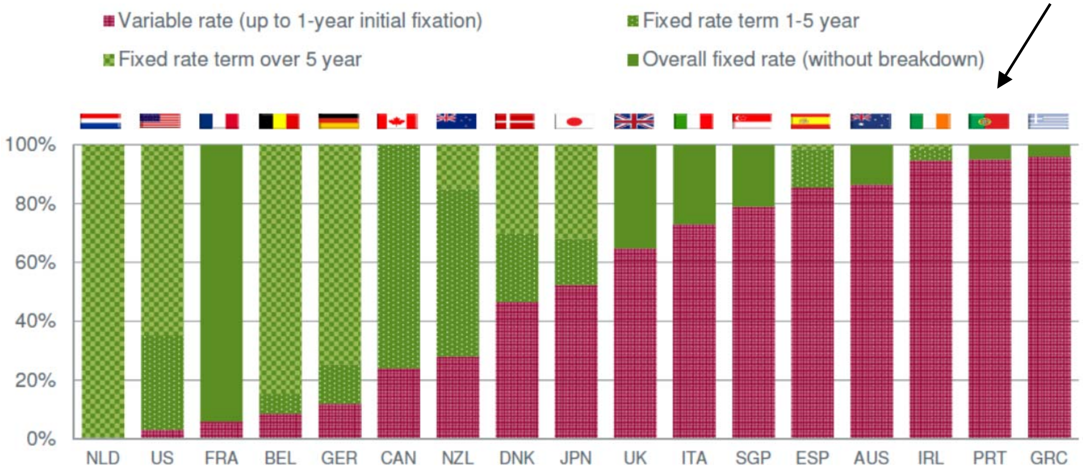
INTEREST RATE RISK

- <u>Sources of interest rate risk</u>:
  - liquidity flows:
    - Direct – new loans, issued debt or deposits received
    - Indirect - prepayments, early redemptions
  - repricing of existing assets and liabilities
- <u>Measurement</u>:
  - <u>interest rate or repricing gaps</u> - corresponding to the differences between the assets and the liabilities to be repriced in different time buckets (usually up to 1 year), excluding non-interest rate bearing balance sheet items (e.g. fixed assets and capital, even though capital may be considered as a fixed rate liability). As in liquidity risk, these gaps may be static or dynamic.
  - <u>difference between average repricing term of assets and liabilities</u> (with fixed rates, corresponds to the differences between residual maturities).

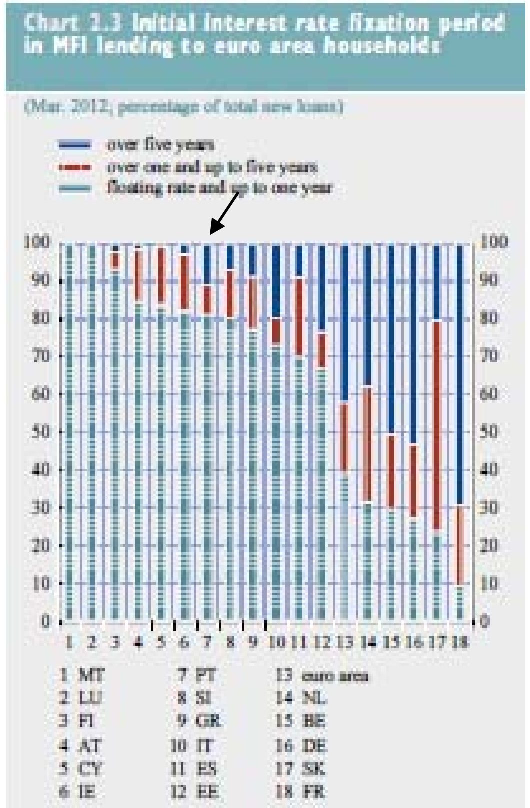
INTEREST RATE RISK

- The goal is to measure the sensitivity of the balance sheet and the P&L to interest rate shifts.
- Hedging of gaps is done through the spot market, forward/futures, options or swaps, as well as by changing the pricing structure of balance sheet.
- Portuguese banks usually have positive interest rate gaps, as credit rates are mostly indexed to money market rates (e.g. Euribor), while among liabilities only bonds issued are usually indexed, given that term deposits are typically short term liabilities (though may be renewed) and their interest rates are typically fixed by the bank.
- Therefore, short term interest rate decreases are, *ceteris paribus*, unfavorable to banks (as long as repricing gaps are shorter for assets than for liabilities).
- However, we must also bear in mind that higher interest rates may reduce credit risk.

INTEREST RATE RISK



Source: Fitch (2016), 2016 Fitch Credit Outlook Conference, Lisbon, 28 Jan.



Source: European Central Bank (2012), "Financial Stability Review", June.

- **Interest rate risk** management creates a demand for mathematical models:
  - Calculation of durations/modified durations for bonds
  - Pricing of exotic options
  - Prepayment models for loans and deposits in banks
  - Estimation of the Term Structure of Interest Rates

1 - Definitions and Notation

- We start with an overview of various concepts of interest rates.

- We also describe the most important financial contracts related to interest rates, and markets at which they are traded.

- We address the main questions concerning interest rate models.

1.1 - ZERO COUPON (OR DISCOUNT) BOND

- **Definition**: By a **zero-coupon bond (a discount bond)** of maturity *T* we mean a financial security paying to its holder one unit of cash at a single pre-specified date *T* in the future, i.e., with no intermediary cash-flows (coupons):

$$
\begin{aligned}
p(t,T) &= \text{price, at } t, \text{ of a } T\text{-bond.} \\
p(T,T) &= 1.
\end{aligned}
$$

- This means that, by convention, the bond's *principal* (also known as the *face value* or *nominal value*) is 1 monetary unit.
- It also means that the price of the bond will tend to one along time (pull-to-par) and at the maturity date it will be 1.

1.2 - FROM BONDS TO INTEREST RATES

Q(S)*P(S)=Q(T)*P(T)
1*p(t,S)=[p(t,S)/p(t,T)]*p(t,T)
p(t,S)=p(t,S)

At time t:

- Sell one S-bond
- Buy exactly $p(t,S)/p(t,T)$ $T-$bonds
- Net investment at t: 0$.

At time S:

- Pay 1$

(the principal of the bond issued, may be funded again by the bond markets)

At time T:

- Collect $p(t,S)/p(t,T) \cdot 1\$$

Net Effect

- The contract is made at t.
- An investment of 1 at time S has yielded $p(t,S)/p(t,T)$ at time T.
- The equivalent constant rates, R, are given as the solutions to

Continuous rate:

$$e^{R\cdot(T-S)}\cdot 1 = \frac{p(t,S)}{p(t,T)}$$

The investment of 1 at time S (the amount that was paid at S) will allow to obtain the return p(t,S)/p(t,T)

Simple rate:

$$[1 + R\cdot(T-S)]\cdot 1 = \frac{p(t,S)}{p(t,T)}$$

1.3 - CONTINUOUS AND SIMPLE INTEREST RATES

- **However, as we are at *t* and *S* is a future moment, *R* is not a spot, but a forward rate, obtained solving the continuous rate formula in the previous slide, in order to *R*:**

1. The **forward rate for the period** $[S,T]$, **contracted at** t is defined by

$$R(t;S,T) = -\frac{\log p(t,T) - \log p(t,S)}{T-S}.$$

2. The **spot rate**, $R(S,T)$, for the period $[S,T]$ is defined by

$$R(S,T) = R(S;S,T).$$

3. **The instantaneous forward rate is the forward for a very short time to maturity, obtained by calculating the limit of the continuous rate formula in the previous slide, when S is very close to T:**

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T} = \lim_{S \to T} R(t;S,T).$$

(when S is very close to T, this corresponds to the definition of derivative)

4. **The instantaneous short rate at t corresponds to the instantaneous forward when T is very close to t, i.e. when the maturity of the forward is close to zero.**

$$r(t) = f(t,t).$$

1.3 – CONTINUOUS AND SIMPLE INTEREST RATES

1. The **simple forward rate L(t;S,T) for the period** $[S,T]$**, contracted at** t is defined by

Solving the last equation of slide 14 in order to R

$$L(t;S,T) = \frac{1}{T-S} \cdot \frac{p(t,S) - p(t,T)}{p(t,T)}$$

2. The **simple spot rate,** $L(S,T)$, for the period $[S,T]$ is defined by

$$L(S,T) = \frac{1}{T-S} \cdot \frac{1 - p(S,T)}{p(S,T)}$$

Given that p(S,S) =1

1.3 – CONTINUOUS AND SIMPLE INTEREST RATES

The **simple spot rate**, $L(T, T+\delta)$, for the period $[T, T+\delta]$ is given by

$$L = \frac{1}{\delta} \cdot \frac{1-p}{p}$$

As T-S in the last equation now becomes δ and *p(S,T)* becomes *p*

1.4 - Coupon-bearing Bonds

- Coupon rate is the stated interest rate on a security
  - It is called the coupon rate because in the past bondholders kept coupons that had to be presented at the payment agents for interest payments
  - It is referred to as an annual percentage of face value
  - It is often paid twice a year
- The **current yield** gives you a first idea of the return on a bond

$$y_c = \frac{c}{P}$$

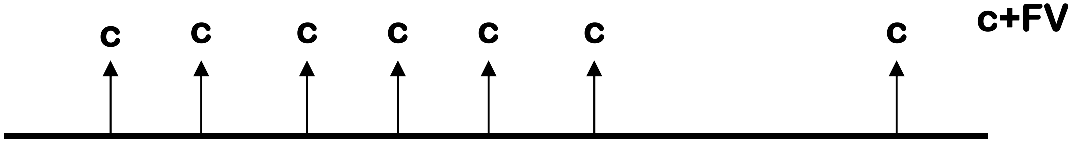
- Example
  - A $1,000 bond has a coupon rate of 7 percent
  - If you buy the bond for $900, your actual current yield is

$$y_c = \frac{70}{900} = 7.78\%$$

1.5 - YIELD-TO-MATURITY

- The **yield-to-maturity** (**YTM**) is the interest rate that makes the present value of the bond's payments equal to its price.

- YTM is the IRR of the cash-flows delivered by the bonds:
  - There is no closed formula to compute the YTM, given that it usually involves equations of order higher than 2.
  - However, YTM may be easily computed by iterative (trial-and-error) methodologies.
  - One of the conceptual problems of YTM is that each cash-flow is discounted using the same rate, implicitly assuming that the yield curve is flat.
  - Conversely, we may be discounting cash-flows of different bonds occurring at the same dates with different discount rates (yields).
  - The yield is equal to the coupon rate whenever the bond price corresponds to the redemption value.



- It is the solution to:
  - Simple YTM (annual)

$$P^c = \frac{FV}{(1+y)^T} + \sum_{n=1}^{T} \frac{c}{(1+y)^n}$$

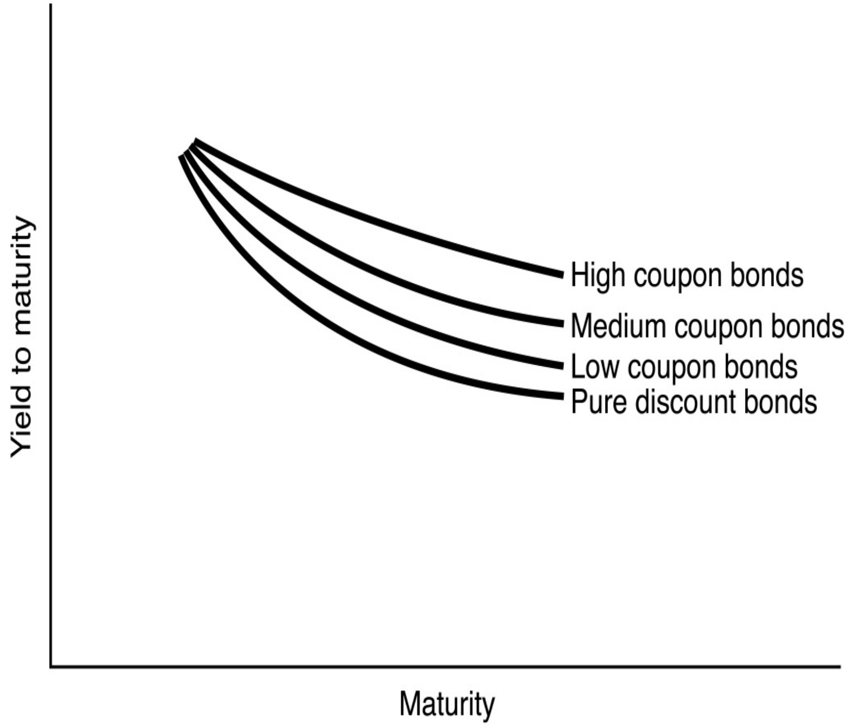
(other)

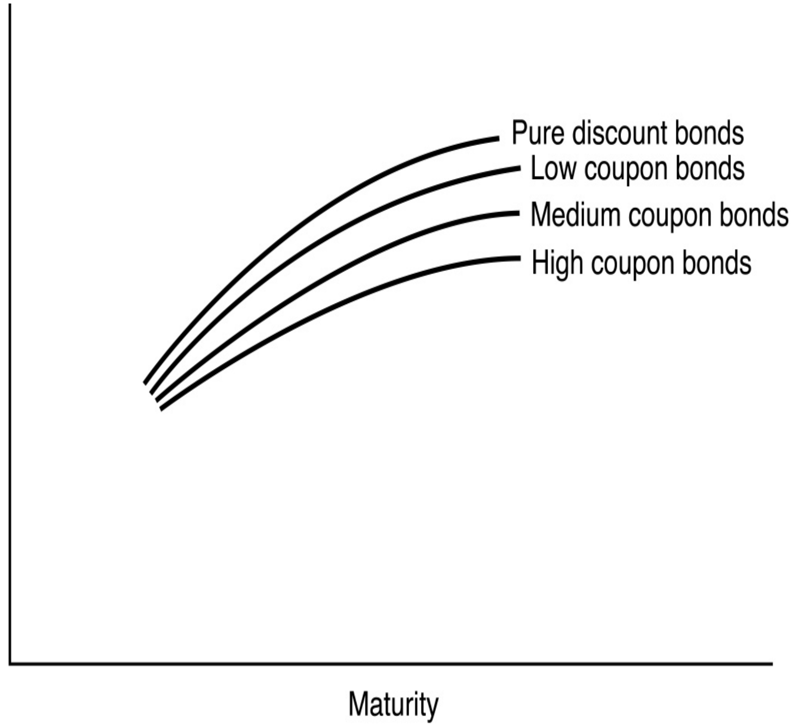
$$P^c = \frac{FV}{\left(1+\frac{y}{m}\right)^{mT}} + \sum_{n=1}^{mT} \frac{c}{\left(1+\frac{y}{m}\right)^n}$$

  - Continuous Time YTM

$$P^c = FVe^{-yT} + \sum_{n=1}^{mT} ce^{-y\left(\frac{n}{m}\right)}$$

- Therefore, the YTM depends on:
  - The bond's maturity
  - The bond's coupon rate
- Actually, the yield curve has a less pronounced shape and curvature when the coupons are higher, as the higher weight of intermediate cash-flows reduces the effect of higher maturities in the discount process.





2 - TERM STRUCTURES (TS)

Empirical Evidence and Classical Theories

2.1. Types of TS

2.2. Dynamics of the TS

2.3. Stylized Facts

2.4. Theories of the TS

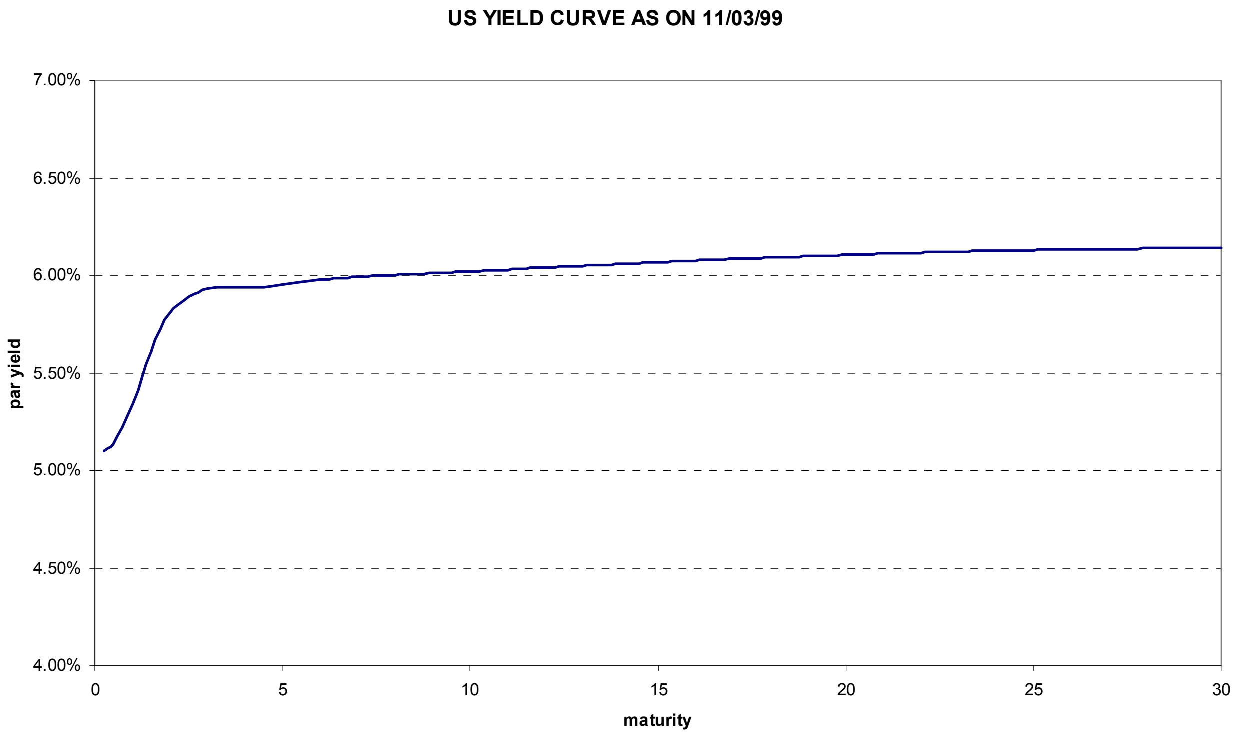
2.1 - TYPES OF TS

- The **term structure of interest rates (TSIR)** is the relationship between spot interest rates and maturities at a given time.

- The TSIR is graphically characterized by the yield curve and may be represented in 3 ways:
  - The spot curve
  - The forward curve
  - The discount factor curve

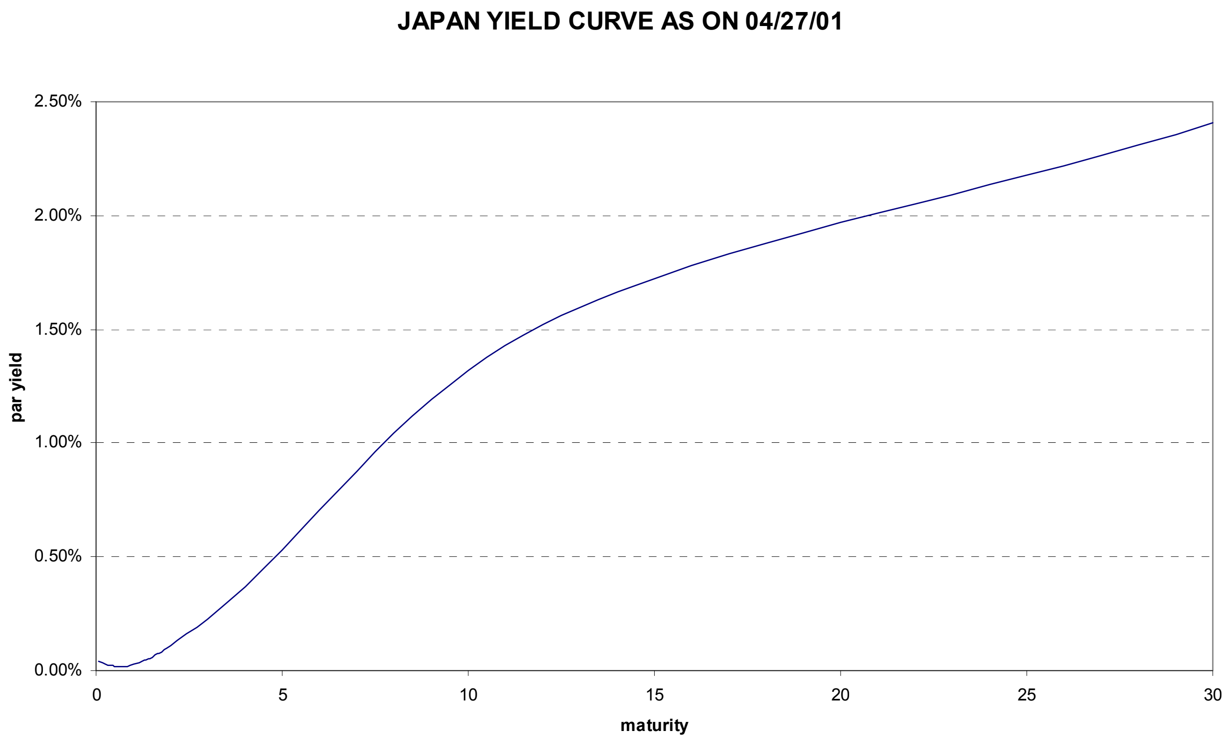
- The yield curve may assume different monotonic or non-monotonic shapes:
  - Quasi-flat
  - Increasing
  - Decreasing
  - Humped

ILLUSTRATION: QUASI-FLAT



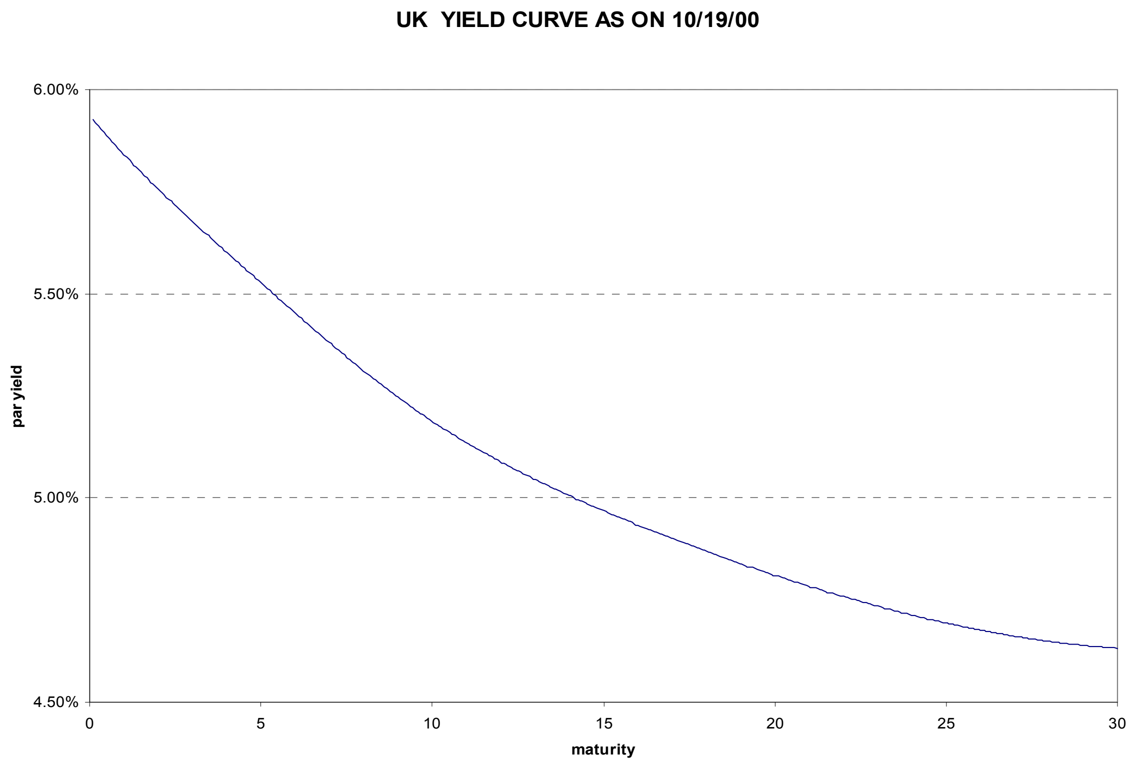
Quasi-Flat

ILLUSTRATION: INCREASING



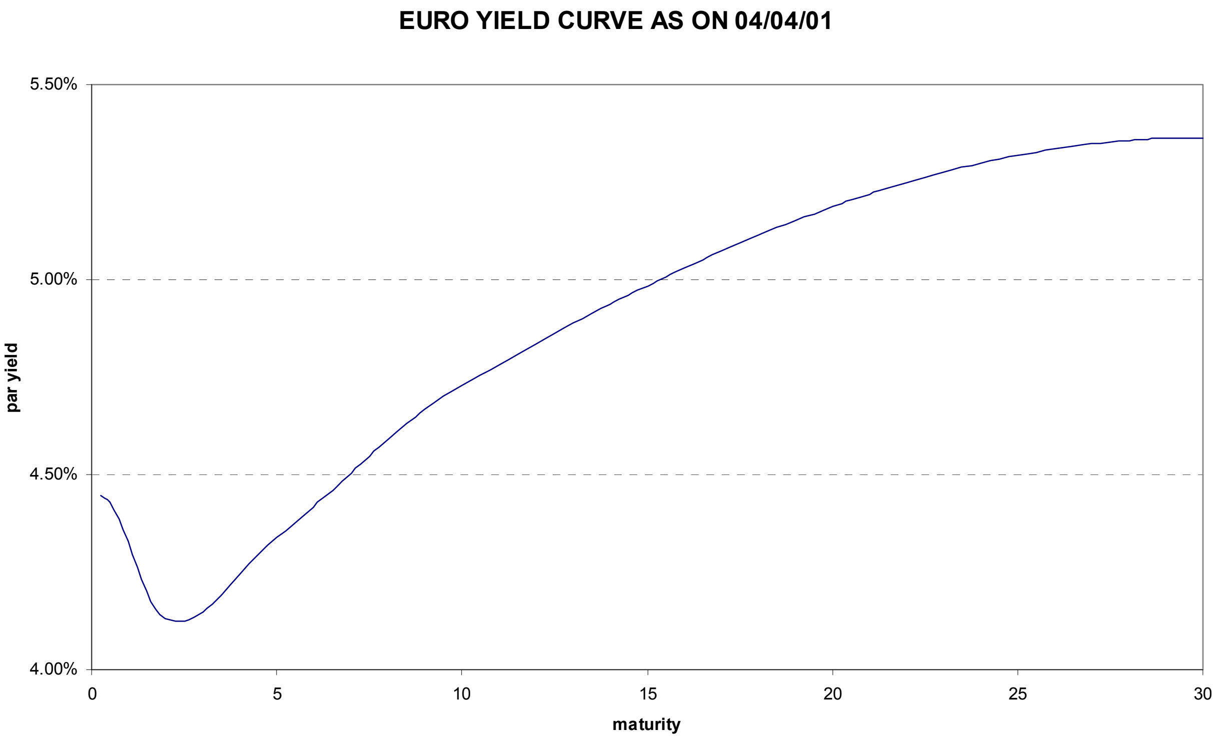
Increasing

ILLUSTRATION: DECREASING



Decreasing (or inverted)

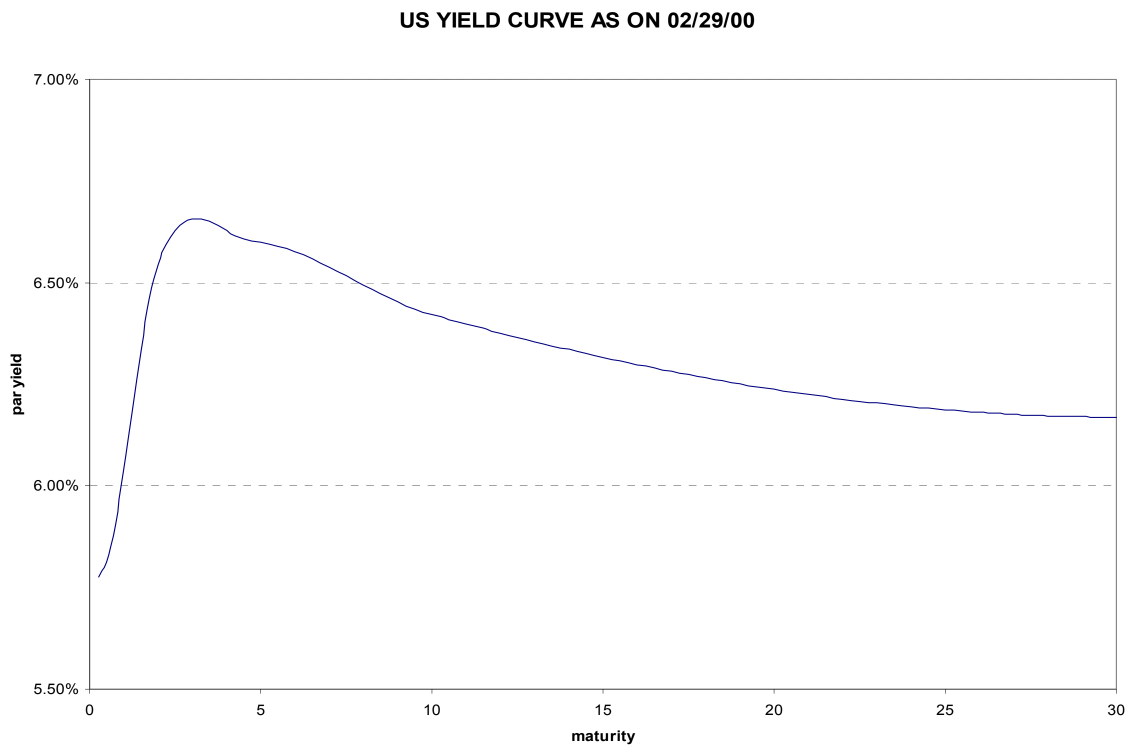
ILLUSTRATION: HUMPED (1)



Humped

(decreasing then increasing)

ILLUSTRATION: HUMPED (2)



Humped

(increasing then decreasing)

2.2 - DYNAMICS OF THE TS

- The term structure of interest rates changes in response to
  - Economic shocks
  - Market-specific events
  - Policy decisions

- Example
  - A Treasury announcement that there will not be any further issuance of 30 year bonds => Price of existing 30 year bonds increases (yield decreases).

2.3 - Stylized Facts

- Volatility
- Correlation
- Standard Movements
  - Shift Movements
  - Twist Movements
  - Butterfly Movements

Stylized Facts (1) : Volatility

- Yields and bond prices are typically much less volatile than prices in other asset classes.
- In countries where the credibility of monetary policy is lower, or, correspondingly, the currency is weaker, the volatility of short term interest rates is usually higher than long term interest rates.
- In countries where credit risk issues arise, long term interest rates typically become more volatile.

STYLIZED FACTS (2) : CORRELATION

- Rates with different maturities are
  - Positively but not perfectly correlated, meaning that there is more than one factor behind the yield curve dynamics
  - Correlation decreases with differences in maturity

- Example:

	1M	3M	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y
1M	1									
3M	0.999	1								
6M	0.908	0.914	1							
1Y	0.546	0.539	0.672	1						
2Y	0.235	0.224	0.31	0.88	1					
3Y	0.246	0.239	0.384	0.808	0.929	1				
4Y	0.209	0.202	0.337	0.742	0.881	0.981	1			
5Y	0.163	0.154	0.255	0.7	0.859	0.936	0.981	1		
7Y	0.107	0.097	0.182	0.617	0.792	0.867	0.927	0.97	1	
10Y	0.073	0.063	0.134	0.549	0.735	0.811	0.871	0.917	0.966	1

STYLIZED FACTS (3): STANDARD MOVEMENTS

- The evolution of the interest rate curve can be split into three standard movements, regardless the time period or the market:
  - **Shift movements** (changes in level), which account for 70 to 80% of observed movements on average.
  - **Twist movements** (changes in slope), which accounts for 15 to 30% of observed movements on average.
  - **Butterfly movements** (changes in curvature), which accounts for 1 to 5% of observed movements on average.

=> One or two factor models tend to be enough to explain the behavior of the yield curve.

2.4 - THEORIES OF THE TS

- Explanatory theories of the TSIR depend mostly on:
  - the preferences of market participants for maturities, i.e their credit and liquidity risk aversion.
  - the expectations on the future behavior of short-term interest rates, i.e. monetary policy
- Term structure theories attempt to explain the relationship between interest rates and their residual maturity.
- Explanatory theories:
  - Expectations
  - Preferred habitat
  - Liquidity premium
  - Market segmentation

- The **expectations theory** postulates that long term rates depend on the current short term rates and the expectations on their future path.
- Actually, let us assume that an investor has 2 investment alternatives:
  - A long term bond (*T* maturity)
  - A set of bonds with short term maturities (maturity = 1, being T>1)
- The expected returns for these 2 alternatives must be equal:
  - $[1+r(t,T)]^n = (1+r(t,1)) \times (1+E(t)\,(r(t+1),1)) \times (1+E(t)\,(r(t+2),1)) \times \ldots \times (1+E(t)\,(r(T-1),1))$
  - $r(t,T)=[(1+r(t,1)) \times (1+E(t)\,(r(t+1),1)) \times (1+E(t)\,(r(t+2),1)) \times \ldots \times (1+E(t)\,(r(T-1),1))]^{1/n}-1$
- If one assumes that there is no risk premium (i.e. investors are risk-neutral regarding investing in short or in long term interest rates), expected interest rates are equal to forward rates.
- According to this theory, the yield curve may assume different shapes and positively (negatively) sloped curves correspond to expectations of short tem interest rate increases (decreases).
- Therefore, changes in yield curves are interpreted as changes in market expectations.

- The **market segmentation theory** postulates that interest rates in each maturity stem only from the supply and demand in that maturity.
- As a consequence, there is no relationship between interest rates in several maturities and the yield curve may have very irregular shapes.
- The **preferred habitat theory** sustains that investors have preferred maturities, but they accept to invest in different maturities if they are compensated for that.
- Therefore, this theory may be seen as a smoothed version of the previous one, with premiums paid to attract investors to maturities different from those preferred, but not necessarily increasing with the maturity.
- Consequently, under this theory, moves in the yield curve do not correspond necessarily to changes in investors' expectations about the future path of short term interest rates and the yield curve may have different shapes.

- In the **liquidity premium theory**, investors always prefer short to long maturities.
- Consequently, long term interest rates will always be higher than short term interest rates, i.e. the yield curve will always be positively sloped.
- Actually, one may conclude that the yield curve shape is explained by a mix of all these theories, even though market participants usually consider that a normal yield curve is a positively sloped one.
- The risk premium is usually considered as increasing with maturities.
- Even though the risk premium is not nil, changes in long-term interest rates may be considered as changes in expectations on future short-term interest rates' behavior if one assumes that risk premium is constant along time – **non-pure (or biased) expectations theory**.

3 – HEDGING INTEREST RATE RISK

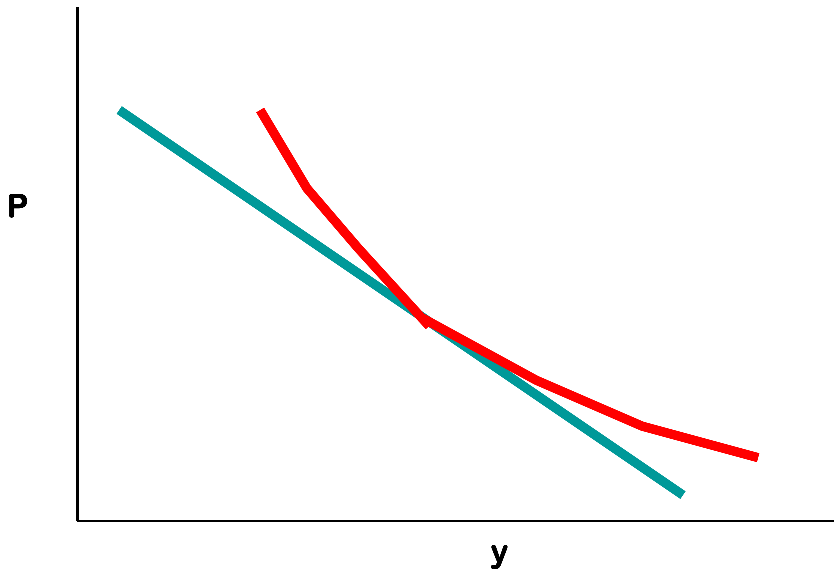
- Basic principle: attempt to reduce as much as possible the dimensionality of the problem, i.e. to hedge risk with as few factors as possible.

- First step: **duration hedging**
  - Consider only one risk factor
  - Assume a flat yield curve
  - Assume only small changes in the risk factor

- Beyond duration
  - Relax the assumption of small interest rate changes: **convexity hedging**
  - Relax the assumption of a flat yield curve: **trivial**
  - Relax the assumption of parallel shifts: **not trivial at all**

3.1 - DURATION HEDGING

- We will study the sensitivity of the bond price to changes in yield - Interest rate risk:
  - Rates change from *y* to *y+dy*
  - *dy* - small variation in yields, e.g. 1 bp (e.g., from 5% to 5.01%)
  - *dP* - variation in bond price due to *dy*
- The relationship between the bond prices and the yields is not linear.
- However, for small changes in yields, a good proxy for *dP* is the first derivative of the bond price in order to *y*.



- With continuously compounded interest rates, we have:

$$\frac{\partial P^c}{\partial y} = \frac{\partial \left[FVe^{-yT_n} + \sum_{n=1}^{n} c_i e^{-yT_i} \right]}{\partial y}$$

$$= -T_n FVe^{-yT_n} - \sum_{n=1}^{n} T_i c_i e^{-yT_i}$$

- **Duration –** (absolute value of) the partial derivative of the bond price with respect to yield, divided by the bond price:

$$D = \frac{\sum_{i=1}^{n} T_i c_i e^{-yT_i} + T_n FVe^{-yTn}}{P^c}$$

  - it tells you how much relative change in price follows a given small change in yield impact

$$\frac{dP^c}{P^c} = -Ddy$$

- With discrete compounding interest rates: (to develop)

$$D = \frac{\sum_{i=1}^{n} \frac{T_i c_i}{(1+y)^{T_i}} + \frac{T_n FV}{(1+y)^{T_n}}}{P}$$

- Its relation to relative change in price is $\frac{dP}{P} \approx -\frac{D}{1+y} dy$
- The ratio $\frac{D}{1+y}$ is called **modified duration**

EXAMPLE (SEE CALCULATION IN SPREADSHEET)

Time of Cash Flow (i)	Cash Flow F_i	$w_i = \frac{1}{P^c} \times \frac{F_i}{(1+y)^i}$	$i \times w_i$
1	53.4	0.0506930	0.0506930
2	53.4	0.0481232	0.0962464
3	53.4	0.0456837	0.1370511
4	53.4	0.0433679	0.1734714
5	53.4	0.0411694	0.2058471
6	53.4	0.0390824	0.2344945
7	53.4	0.0371012	0.2597085
8	53.4	0.0352204	0.2817635
9	53.4	0.0334350	0.3009151
10	1053.4	0.6261237	6.2612374
		Total	8.0014280

T = 10, c = 5.34%, y = 5.34%

$$D = \sum_{i=1}^{m} i \times w_i \cong 8$$

DURATION HEDGING - PROPERTIES

- Duration is the weighted average maturity of all bond cash-flows.
- Duration of a zero coupon bond is equal to its maturity
- For a given maturity and yield, duration increases as coupon rate decreases, given that the relative weight of the final cash-flow increases.
- For a given coupon rate and yield, duration increases as maturity increases.
- For a given maturity and coupon rate, duration increases as yield rate decreases, as the net present value of the cash-flows increase more in longer than in shorter maturities.

DURATION HEDGING - PRINCIPLE

- Principle: immunize the value of a bond portfolio with respect to changes in yield
  - Denote by P the value of the portfolio
  - Denote by H the value of the hedging instrument
- Hedging instrument may be
  - Bonds
  - Swaps
  - Futures
  - Options
  - FRAs
- Duration hedging is very simple to do, but it is only valid for small changes and parallel shifts of the yield curve.

- Changes in value
  - Portfolio $$dP \approx P'(y)dy$$
    - Hedging instrument $$dH \approx H'(y)dy$$
- Strategy: hold *q* units of the hedging instrument so that

$$dP + qdH = \left(qH'(y) + P'(y)\right)dy = 0$$

- Solution

$$q = -\frac{P'(y)}{H'(y)} = -\frac{P \times D_P}{H \times D_H}$$

Example:

- At date t, a portfolio P has a price of \$328635, a 5.143% yield and a 7.108 duration

- Hedging instrument, a bond, has a price of \$118.786, a 4.779% yield and a 5.748 duration

- Hedging strategy involves taking a short position (i.e. selling futures contracts) as follows:

$$q = -\frac{P'(y)}{H'(y)} = -\frac{P \times D_P}{H \times D_H}$$

q = -(328635x7.108)/(118.786x5.748) = - 3421

- Therefore, 3421 units of the hedging bond should be sold.

3.2 - CONVEXITY

- Considering a second order Taylor approximation:

$$\frac{dP}{P} = \underbrace{\frac{dP}{dy}\frac{1}{P}}_{-D}(dy) + \frac{1}{2}\underbrace{\frac{d^2P}{dy^2}\frac{1}{P}}_{C}(dy)^2$$

$$C = \frac{\partial^2 P}{\partial y^2}\frac{1}{P}$$

- With continuous compounding:

$$\frac{\partial^2 P}{\partial y^2} = (T_n)^2 FVNe^{-yT_n} + \sum_{n=1}^{n}(T)_i^2 c_i e^{-yT_i}$$

$$\frac{\partial P^c}{\partial y} = \frac{\partial\left[FVe^{-yT_n} + \sum_{n=1}^{n} c_i e^{-yT_i}\right]}{\partial y}$$

$$= -T_n FVe^{-yT_n} - \sum_{n=1}^{n} T_i c_i e^{-yT_i}$$

Convexity

- Being the Convexity effect $C = \frac{\partial^2 P}{\partial y^2}\frac{1}{P}$

- The convexity will be:

$$C = \frac{(T_n)^2 FVe^{-yT_n} + \sum_{n=1}^{n}(T)_i^2 c_i e^{-yT_i}}{P^c}$$

$$D = \frac{\sum_{i=1}^{n}\frac{T_i c_i}{(1+y)^{T_i}} + \frac{T_n VN}{(1+y)^{T_n}}}{P}$$

- Discrete capitalization

$$C = \frac{\sum_{t=1}^{T}\frac{C_t t(t+1)}{(1+y)^t} + \frac{T(T+1)FV}{(1+y)^T}}{P^c}$$

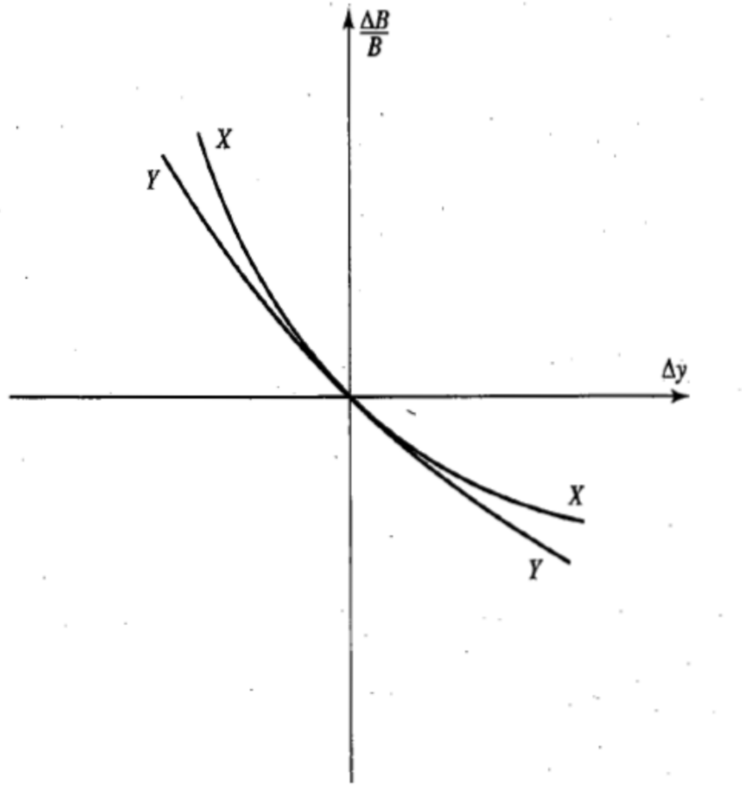
$$\frac{\partial^2 P}{\partial y^2}\Big/P = \underbrace{\frac{1}{(1+y)^2}C}$$

modified convexity

CONVEXITY- PROPERTIES

- Convexity **is always positive** and it is **linear**
- For a given maturity and yield, convexity increases as coupon rate **decreases**
- For a given coupon rate and yield, convexity increases as maturity **increases**
- For a given maturity and coupon rate, convexity increases as yield rate **decreases**
- A bond with higher convexity is always preferred, as its price benefits more from yield decreases and its less impacted by yield increases.

Figure 4.2 Two bond portfolios with the same duration.



Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition

Duration + Convexity Hedging - Principle

- Principle: immunize the value of a bond portfolio with respect to changes in yield
  - Denote by *P* the value of the portfolio
  - Two hedging instruments, whose value is H_1 and H_2, because one wants to hedge one risk factor (parallel shifts in the yield curve) but also the second order effect:
  - Portfolio $$dP \approx P'(y)dy + \frac{P''(y)}{2}dy^2$$
  - Hedging instruments $$\begin{cases} dH_1 \approx H_1'(y)dy + \frac{1}{2}H_1''(y)dy^2 \\ dH_2 \approx H_2'(y)dy + \frac{1}{2}H_2''(y)dy^2 \end{cases}$$

- Strategy: hold q_1 and q_2 units of the first and second hedging instruments so that

$$dP + q_1 \times dH_1 + q_2 \times dH_2 = 0$$

- Solution

  - Under the assumption of unique *dy* – parallel shifts)

$$\begin{cases} P'(y) + q_1 H_1'(y) + q_2 H_2'(y) = 0 \\ P''(y) + q_1 H_1''(y) + q_2 H_2''(y) = 0 \end{cases}$$

  – Or under the assumption of a unique *y* – flat yield curve (in this case, the change in the hedging instruments' and bond prices will result straight from the duration):

$$\begin{cases} q_1 H_1(y) D_1 + q_2 H_2(y) D_2 = -P(y) D_p \\ q_1 H_1(y) C_1 + q_2 H_2(y) C_2 = -P(y) C_p \end{cases}$$

EXAMPLE:

- Portfolio at date t
  - Price P = $ 32863.5
  - Yield y = 5.143%
  - Modified duration = 6.76
  - Convexity Conv = 85.329

- Hedging instrument 1
  - Price H_1 = $ 97.962
  - Yield y_1 = 5.232 %
  - Modified duration = 8.813
  - Convexity $Conv_1$ = 99.081

- Hedging instrument 2:
  - Price H_2 = $ 108.039
  - Yield y_2 = 4.097%
  - Modified duration = 2.704
  - Convexity $Conv_2$ = 10.168

- Optimal quantities q_1 and q_2 of each hedging instrument are given by

$$\begin{cases} q_1 H_1(y) D_1 + q_2 H_2(y) D_2 = -P(y) D_p \\ q_1 H_1(y) C_1 + q_2 H_2(y) C_2 = -P(y) C_p \end{cases}$$

$$\begin{cases} q_1 \times 8.813 \times 97.962 + \\ + q_2 \times 2.704 \times 108.039 = -32863.5 \times 6.76 \\ \\ q_1 \times 99.081 \times 97.962 + \\ + q_2 \times 10.168 \times 108.039 = -32863.5 \times 85.329 \end{cases}$$

- Solving in order to $q1$ and $q2$, you should sell 305 units of H_1 and buy 140 units of H_2.

DURATION + CONVEXITY HEDGING - LIMITATIONS

- Actually, the yield curve is not flat.
- Furthermore, yield curve shifts are not only parallel, but its shape also changes.
- Therefore, the yield curve dynamics is not fully explained by one factor models.

4 –IR DERIVATIVES

- Forward Rate Agreements (**FRAs**)
- Interest rate futures
- Interest sate swaps (**IRS**)
- Interest rate **Options:**
  - Plain vanilla Bond or short-term futures **Options**
  - Interest rate **CAPS**
  - **Swaptions**

FRAs

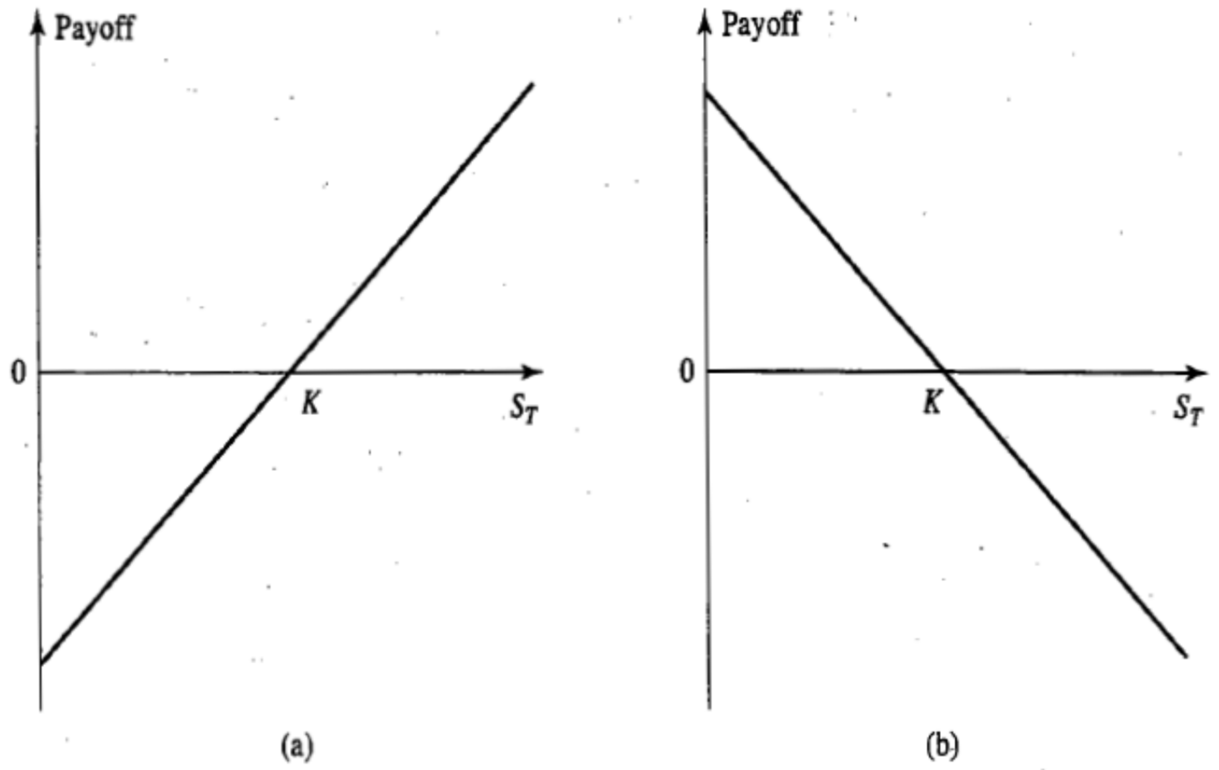
- A **forward rate agreement** FRA is a contract involving three time instants: The current time t, the expiry time $T > t$, and the maturity time $S > T$. The contract gives its holder an interest rate payment for the period $T \mapsto S$ with fixed rate K at maturity S against an interest rate payment over the same period with rate $L(T, S)$.
- Basically, this contract allows one to lock–in the interest rate between T and S at a desired value K.
- *T* is also usually known as the settlement date and the difference between *t* and *T* as the time to settlement.
- If one assumes that investor are risk-neutral, the FRA's interest rate corresponds to the expected interest rate for time *T* and term *S*-*T*.

FRAs

- FRAs are quoted by financial institutions in over-the-counter market.
- Therefore, they are not listed in exchange markets.
- Additionally, they are quoted for fixed times to settlement and maturities, e.g. 3 x 9 (6-month interest rate forward, with a time to settlement of three months).
- Quotes are in percentage points, as usual interest rates (e.g. 2%, 4%).
- In order to cancel an exposure to a FRA, one cannot sell the contract, but can take a short exposure on a futures contract, for a time to settlement and an interest rate maturity as close as possible.

FRAs

Figure 1.2 Payoffs from forward contracts: (a) long position, (b) short position. Delivery price = K; price of asset at contract maturity = S_T.



Source: Hull, John (2009), "Options, Futures and Other Derivatives", Pearson Prenctice Hall, 7th Edition

INTEREST RATE FUTURES

- Futures contracts are traded in exchange markets, for fixed settlement dates (and consequently different times to maturity, e.g. the 3-month Euribor futures for Dec.2016 settlement).
- Therefore, in order to cancel a long position in such a contract, an investor can take a short position in (sell) the same contract.
- Usually, one can find futures contracts for short and long term interest rates.
- Short-term futures contracts are quoted as 100 minus the implied interest rates.
- Long-term futures contracts are priced as a % of the nominal value of the theoretical bond embedded in the futures contract, just like the true bonds.

INTEREST RATE FUTURES

- Therefore, an increase in both types of futures contracts means that the implied interest rate is decreasing.

- Short and long term interest rate futures are usually available for quarterly settlement dates (pre-specified days in March, June, September and December).

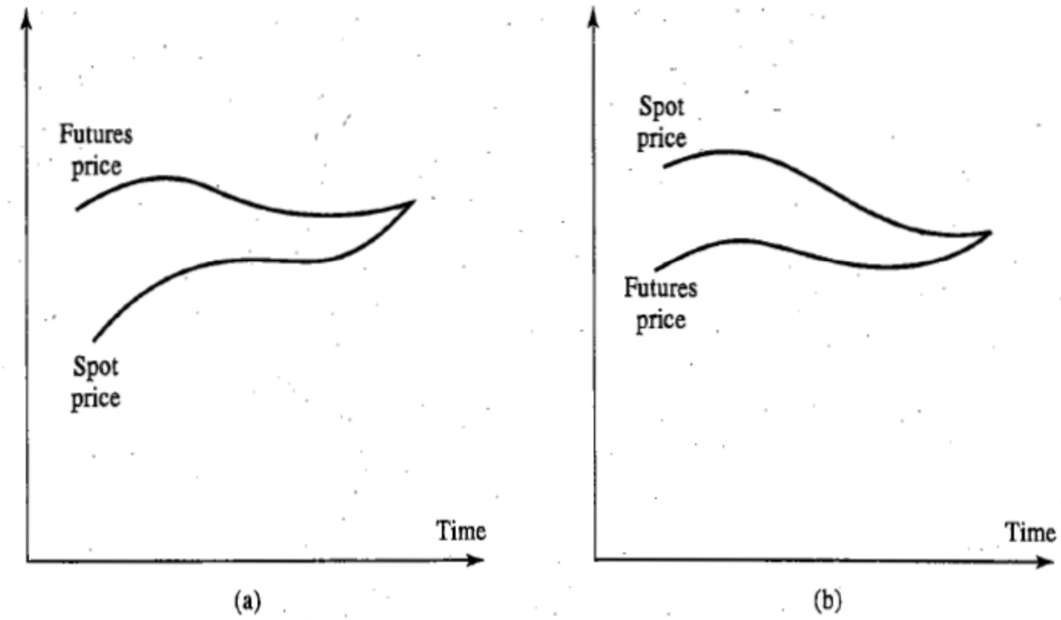
- However, short term futures are available for a longer set of settlement dates, event though the most liquid contracts are those with shorter times to settlement.

- Short-term interest rate futures have financial settlement, while long-term interest rate futures usually have physical settlement, through the cheapest-to-deliver bond.

INTEREST RATE FUTURES

•Futures prices converge to the spot value when the settlement date is approaching.

Figure 2.1 Relationship between futures price and spot price as the delivery period is approached: (a) Futures price above spot price; (b) futures price below spot price.



Source: Hull, John (2009), “Options, Futures and Other Derivatives”, Pearson Prenctice Hall, 7th Edition

IRS

- IRS are contracts that settle the exchange of fixed for variable interest rates at pre-specified dates.

- Therefore, they may be seen as a long (short) position in a fixed rate bond or a set of FRAs, on one hand, and a short (long) position in a variable interest rate bond, on the other hand.

- The swap value or price corresponds to its replacement cost, i.e. the amount of money that should be paid to one counterparty to the other to cancel the contract, reflecting the dynamics of short and long term interest rates since the initial date or the last payment date.

- This also corresponds to the difference between a fixed and a floating rate bond:

(from the floating rate payer's point of view) $V_{\text{swap}} = B_{\text{fix}} - B_{\text{fl}}$

- Consequently, at the initial and all payment dates, the swap value returns to 0.

INTEREST RATE OPTIONS

• Interest rate volatilities have to be estimated.

• Consequently, the Dynamics of the yield curve must be assessed => Stochastic Interest Rate Models.

• A **cap** is a set of put options on the several interest rates to be paid in the future.

• Each of these options is a **caplet** and can be traded individually.

• A **swaption** is an option giving the right to enter into a swap at a future pre-specified rate.

• Therefore, the option will be exercised only if the IRS has a positive value.

• Contrary to the caps, the several cash-flows of the swaption cannot be traded separately and the whole yield curve has to be modelled, including the correlation between the several interest rates.

Problem:

We want to price, at t, a European Call, with exercise date S, and strike price K, on an underlying T-bond. $(t < S < T)$.

Naive approach: Use Black-Scholes's formula.

$$F(t,p) = pN[d_1] - e^{-r(S-t)}KN[d_2].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{S-t}}\left\{\ln\left(\frac{p}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(S-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{S-t}. \end{aligned}$$

where

$$p = p(t,T)$$

Is this allowed?

- p shall be the price of a traded asset. OK!
- The volatility of p must be constant. Here we have a problem because of **pull-to-par**, i.e. the fact that $p(T, T) = 1$. Bond volatilities will tend to zero as the bond approaches the time of maturity.
- The short rate must be **constant** and **deterministic**. Here the approach collapses completely, since the whole point of studying bond prices lies in the fact that interest rates are stochastic.

- Therefore, one needs to estimate a model for bond prices or interest rates.